Metodo semplificato per la misura con radioisotopi di alcuni effetti isotopici in cinetica chimica.

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Riassunto. — Viene presentato ed analizzato dal punto di vista degli errori un metodo per la misura con radioisotopi degli effetti isotopici sulle costanti di velocità di reazioni di equilibrio del secondo ordine.

La conoscenza degli effetti isotopici in cinetica chimica presenta un notevole interesse sia come guida all'utilizzazione pratica dei fenomeni di frazionamento isotopico, sia come metodo di controllo delle teorie del meccanismo delle reazioni chimiche. Per quest'ultimo scopo si presenta particolarmente indicato lo studio degli effetti isotopici in relazioni omogenee decorrenti con meccanismi piuttosto semplici, tali cioè da poter essere trattate teoricamente con l'introduzione del minor numero possibile di ipotesi arbitrarie. Tali sono alcune reazioni termiche di equilibrio che procedono con meccanismo bimolecolare (come, ad esempio, la formazione di acido iodidrico da iodio ed idrogeno), rappresentabili schematicamente con l'equazione chimica

$$A_2 + B_2 \leftrightarrow 2AB ,$$
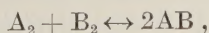
delle quali viene trattato nel presente lavoro.

La determinazione degli effetti isotopici è in generale non molto agevole quando si voglia raggiungere la precisione richiesta per un utile confronto tra

i risultati sperimentali e quelli teorici. Ciò non soltanto per la piccola entità degli effetti stessi, ma anche perchè, ove si voglia eseguire la misura operando separatamente con le singole specie isotopiche, si va incontro, oltrechè alla necessità di disporre di sostanze isotopicamente pure, anche alla impossibilità di usare nuclidi radioattivi (ad eccezione di quelli a vita media estremamente lunga), dato che la radiazione necessariamente presente modificherebbe profondamente il meccanismo termico della reazione.

Ove invece si voglia operare con specie isotopiche miscelate, è possibile eseguire a priori una valutazione approssimata della condizione necessaria affinchè il contributo della radiazione al decorso della reazione si mantenga molto piccolo. Una valutazione media molto grossolana di tale condizione (eseguita assumendo un valore di G — e cioè del numero di molecole AB formate per 100 eV perduti dalla radiazione — di alcune unità ed un valore dell'energia della radiazione di 10^5 eV) conduce all'espressione

$$(1) \qquad\qquad bk\tau \ll 10^4 \, \frac{a^*}{a + a^*} \,,$$

dove con b si è indicata la concentrazione del reagente B_2, con k la costante di velocità della reazione, con τ la vita media della specie isotopica radioattiva presente nel reagente A_2, con a^* la sua concentrazione e con a quella della specie stabile. Come si vede, già per valori di G non molto elevati, per $a^*/(a+a^*) \simeq 1$ l'isotopo radioattivo deve avere vita media dell'ordine delle migliaia di anni affinchè, in reazioni che si svolgono con velocità non eccessive — e cioè tali da consentire misure sufficientemente accurate $(bk \leqslant 2 \cdot 10^{-5} \, \text{s}^{-1})$ —, il contributo della radiazione alla reazione si mantenga dell'ordine delle unità per mille. È però possibile, data la estrema sensibilità dei metodi di misura della radioattività, assegnare ad $a^*/(a + a^*)$ valori anche molto piccoli ed utilizzare perciò (a parte altre difficoltà) qualsiasi isotopo radioattivo. Con l'impiego di isotopi miscelati, d'altra parte, la trattazione matematica del problema diviene alquanto ardua.

In effetti, se nella reazione in esame il reagente A_2 è costituito da una miscela di due isotopi A ed A*, si avranno in realtà contemporaneamente le tre reazioni:

$$A_2 + B_2 \leftrightarrow 2AB \,,$$

$$AA^* + B_2 \leftrightarrow AB + A^*B \,,$$

$$A_2^* + B_2 \leftrightarrow 2A^*B \,.$$
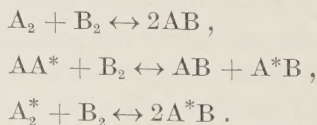
Una prima semplificazione del problema si ottiene immediatamente considerando che per i valori di $a^*/(a + a^*)$ che rendono soddisfatta la condizione (1) si avrà sempre $[A_2^*] \ll [AA^*]$ e che pertanto il contributo della terza reazione potrà essere trascurato senza commettere errori apprezzabili nella determinazione delle costanti di velocità delle altre due

Indicando quindi con a, a^*, b, c e c^* le concentrazioni iniziali di A_2, AA^*, B_2, AB e A^*B, con x ed x^* le concentrazioni di A_2 ed AA^* all'istante t, con k e k^* le costanti di velocità, con E ed E^* le costanti di equilibrio, il decorso simultaneo delle due reazioni può essere descritto dal seguente sistema:

$$(2) \quad \begin{cases} -\dfrac{1}{k}\dfrac{dx}{dt} = x[b - (a-x) - (a^* - x^*)] - \dfrac{1}{E}\,[c + 2(a-x) + (a^* - x^*)]^2\,, \\[2ex] -\dfrac{1}{k^*}\dfrac{dx^*}{dt} = x^*[b - (a-x) - (a^* - x^*)] - \\[2ex] \qquad\qquad - \dfrac{1}{E^*}\,[c + 2(a-x) + (a^* + x^*)]\,[c^* + (a^* - x^*)]\,. \end{cases}$$

Questo sistema di equazioni differenziali non lineari non è riconducibile alle quadrature; esso è riconducibile all'equazione differenziale del secondo ordine detta di ABEL [1], integrabile in particolari condizioni, che però non si verificano nel presente caso per nessuna scelta dei valori delle costanti iniziali. Sarebbe quindi necessario ricorrere a metodi di integrazione approssimata, tenendo presente che ciò che interessa ricavare è il rapporto k/k^* od anche la sola k^*, essendo l'altra costante misurabile in modo indipendente. L'applicazione dei metodi generali di integrazione approssimata conduce però ad espressioni matematiche così complesse da non permettere di ricavare esplicitamente nè le costanti di velocità nè il loro rapporto.

È però possibile ricercare particolari condizioni sperimentali che consentano di sostituire i secondi membri delle (2) con opportune funzioni di x ed x^*, tali da permettere di integrare il nuovo sistema e di ottenere espressioni che diano valori sufficientemente approssimati di k e k^* o del loro rapporto. L'entità della approssimazione che si raggiunge operando una simile sostituzione non è in generale calcolabile con facilità in maniera esatta, ma può essere valutata in maniera sufficiente a fornire l'ordine di grandezza dell'errore. Si può mostrare infatti, sia pure con un ragionamento grossolano, che, se si indica con δ la massima differenza fra l'espressione esatta e quella approssimata dei secondi membri delle (2), è

$$(3) \qquad \frac{\Delta k}{k} \simeq \frac{\Delta k^*}{k^*} \ll \delta \,.$$

Uno di tali casi in cui le condizioni sperimentali si presentano particolarmente favorevoli per operare una sostituzione semplificativa nei secondi membri delle (2), si ha quando le costanti di equilibrio E ed E^* hanno valori così elevati che il secondo termine è trascurabile rispetto al primo anche per valori di $a - x$ ed $a^* - x^*$ abbastanza grandi per consentire misure sufficientemente precise.

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[1] E. KAMKE: *Differentialgleichungen*, Vol. I, p. 26.

Per l'ordine di grandezza di E ed E^*, che consente mediante questa approssimazione determinazioni di k/k^* precise a qualche unità per cento, si ottiene mediante la (3) il valore di 10^2. Si deve inoltre tener presente che esiste una condizione anche per il valore della costante di velocità, affinchè la durata dei transitori iniziale e finale della reazione rappresentino una frazione abbastanza piccola del tempo totale.

Un altro caso che può presentarsi particolarmente favorevole è quello qui proposto: il metodo esaminato consente la determinazione della costante k^* utilizzando la peculiarità dei metodi di determinazione radiochimica, che rendono possibili misure sufficientemente precise di rapporti tra concentrazioni anche estremamente piccole di radioisotopi. Per tale ragione, infatti, è possibile eseguire la reazione in questione in condizioni nelle quali le (2) subiscono una sostanziale semplificazione.

A tale scopo una certa quantità del composto AB (esente da tracce di A^*B) alla concentrazione c viene riscaldata alla temperatura alla quale si intende determinare k^*, finchè viene raggiunto l'equilibrio tra A_2, B_2 ed AB. A questo punto viene introdotta nel recipiente una certa quantità di una miscela $A + A^*$, la cui concentrazione in A^*, non necessariamente nota con esattezza, deve essere tale da escludere praticamente la presenza di molecole A_2^*. La quantità di tale miscela $A_2 + AA^*$ introdotta nel recipiente di reazione deve essere così piccola da non modificare in maniera apprezzabile la concentrazione di A_2. Per contro la concentrazione iniziale di AA^*, a^*, che viene così a stabilirsi nel recipiente deve essere sufficientemente grande per poter essere misurata con la necessaria precisione. Si trova che queste due esigenze possono essere entrambe soddisfatte, se si considera che con nuclidi radioattivi con tempi di dimezzamento minori di 10^4 anni sono sufficienti per il secondo scopo di concentrazioni relative di AA^* (rispetto ad A_2) minori di $10^{-5} \div 10^{-6}$. Operando quindi con gli accorgimenti detti, le reazioni in studio potranno essere descritte con ottima approssimazione dalle seguenti equazioni:

$$(4) \quad \begin{cases} 0 = x^2 - \dfrac{1}{E}(c - 2x)^2 \,, \\[3mm] -\dfrac{1}{k^*}\dfrac{\mathrm{d}x^*}{\mathrm{d}t} = Xx^* - \dfrac{1}{E^*}(c - 2X)(a^* - x^*) \,, \end{cases}$$

dove con X si è indicato la radice positiva della prima equazione. Ponendo $x^*/a^* = \varrho$ ed operando opportune sostituzioni, si può riscrivere la precedente equazione differenziale nella forma:

$$-\frac{E^*}{E^* + \sqrt{E}}\,\frac{2 + \sqrt{E}}{c}\,\frac{1}{k^*}\,\frac{\mathrm{d}\varrho}{\mathrm{d}t} = \varrho - \frac{\sqrt{E}}{E^* + \sqrt{E}} \,.$$

È facile controllare che il secondo termine del secondo membro rappresenta il valore di ϱ all'equilibrio, cioè il valore minimo, ϱ_m, raggiungibile da questa grandezza. Si potrà allora scrivere:

$$- (1 - \varrho_m) \frac{2 + \sqrt{E}}{c} \frac{1}{k^*} \frac{d\varrho}{dt} = \varrho - \varrho_m .$$

Integrando fra 0 e t si ottiene immediatamente:

$$(5) \qquad k^* = \frac{1 - \varrho_m}{t} \frac{2 + \sqrt{E}}{c} \log \frac{1 - \varrho_m}{\varrho - \varrho_m} .$$

Conviene osservare che le grandezze che figurano in questa espressione oltre a c, E e t sono tutti rapporti tra concentrazioni dell'isotopo radioattivo e che pertanto tale metodo di misura della costante di velocità k^* presenta il vantaggio di non comportare nessuna misura assoluta di radioattività specifica. Tale metodo non presenta quindi difficoltà pratiche maggiori di quelle offerte dall'impiego di isotopi stabili puri; esso anzi evita l'uso di isotopi carrier-free e consente di estendere la misura degli effetti isotopici ai nuclidi radioattivi anche nella importante categoria delle reazioni bimolecolari di equilibrio termico.

L'essenza dell'artificio proposto, d'altronde abbastanza evidente, consiste nel far decorrere la reazione

$$AA^* + B_2 \leftrightarrow AB + A^*B ,$$
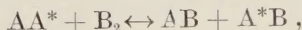
in presenza di quantità relativamente grandi di A_2, B_2 ed AB in equilibrio chimico, in condizioni cioè in cui essa procede essenzialmente come se evolvesse da sola, senza peraltro andare incontro nè agli inconvenienti derivanti dalla intensa radiazione di isotopi carrier-free, nè alla necessità di conoscere la composizione isotopica della miscela impiegata.

È interessante osservare che attraverso le medesime misure può essere ricavata anche la costante di equilibrio E^*, che, come si vede immediatamente, è data da

$$E^* = \sqrt{E} \frac{1 - \varrho_m}{\varrho_m} .$$

Ricordando quanto detto sopra sull'importanza di ottenere determinazioni delle costanti di velocità con precisioni molto spinte, appare opportuno analizzare la (5) dal punto di vista degli errori, per vedere in che misura l'incertezza nelle determinazioni sperimentali delle grandezze che vi figurano si ripercuota sulla precisione con cui è ottenibile il k^*. Vengono qui esaminate

separatamente e nell'ordine le incertezze introdotte nella misura di k^* dagli errori sperimentali su E, ϱ e ϱ_m, essendo il caso di c e t del tutto ovvio.

Per ciò che riguarda E, si avrà:

$$\frac{\Delta k^*}{k^*} = \frac{\sqrt{E}}{2(2 + \sqrt{E})} \frac{\Delta E}{E} .$$

Come si vede la situazione è molto favorevole, in quanto il coefficiente per cui bisogna moltiplicare l'errore percentuale sulla E per avere il corrispondente errore sulla k^*, risulta sempre inferiore all'unità.

Da un analogo esame dell'incertezza prodotta sul k^* dagli errori di misura di ϱ si ha:

$$\frac{\Delta k^*}{k^*} = \frac{\varrho}{\varrho - \varrho_m} \frac{1}{\log \dfrac{1 - \varrho_m}{\varrho - \varrho_m}} \frac{\Delta \varrho}{\varrho} .$$

Si vede che sarà opportuno scegliere convenienti condizioni di misura del ϱ, tali cioè che il coefficiente di $(\Delta\varrho)/\varrho$ divenga minimo. Questa condizione è verificata quando

$$\log \frac{\varrho - \varrho_m}{1 - \varrho_m} = - \frac{\varrho}{\varrho_m} .$$

Tale equazione trascendente in ϱ può essere risolta in via approssimata tenendo conto che $\varrho_m < \varrho < 1$ e che ϱ_m ha, di norma, il valore di qualche unità per 10^{-1}. Sviluppando allora $\log (\varrho - \varrho_m)/\varrho_m(1 - \varrho_m))$ secondo $\log x = (x - 1) - \frac{1}{2}(x - 1)^2$, si ottiene

$$(6) \qquad \varrho = (\beta - \sqrt{\beta^2 - \gamma}) ,$$

dove $\beta = (2 - \varrho_m)^2$ e $\gamma = 4(1 - \varrho_m) + 1 + (1 - \varrho_m)^2(3 - 2 \log \varrho_m)$. Per un dato ϱ_m il coefficiente dell'errore relativo su ϱ presenta un minimo per il valore di ϱ dato dalla (6). Tali valori ottimi di ϱ in funzione di ϱ_m sono rappresentati dalla curva A di Fig. 1. Nella stessa Figura la curva B rappresenta invece l'andamento, in funzione di ϱ_m, del rap-
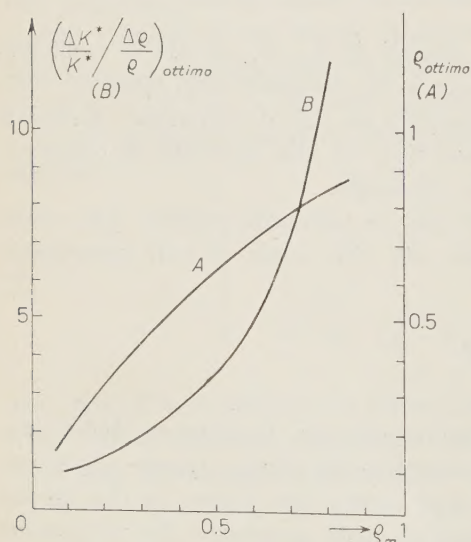

Fig. 1.

porto, calcolato nelle condizioni di ottimo, fra l'errore percentuale su k^* e l'errore percentuale su ϱ. Come si vede tale rapporto mantiene valori accettabili per $\varrho_m \leqslant 0.5$, mentre per valori di ϱ_m più elevati, e cioè per condizioni di equilibrio fortemente spostato verso la dissociazione del composto AB, assume valori tali da rendere eccessivamente critica la misura di ϱ.

Gli errori di misura di ϱ_m si ripercuotono su k^* secondo l'espressione:

$$\frac{\Delta k^*}{k^*} = \frac{\varrho_m}{1-\varrho_m}\left(\frac{1}{\log\dfrac{1-\varrho_m}{\varrho-\varrho_m}}\,\frac{1-\varrho}{\varrho-\varrho_m} - 1 \right)\frac{\Delta\varrho_m}{\varrho_m}.$$

Si vede subito che per ogni ϱ_m le condizioni di ottimo del coefficiente $(\Delta k^*/k^*)/(\Delta\varrho_m/\varrho_m)$ si hanno per ϱ tendente ad 1, mentre al tendere di ϱ verso ϱ_m tale coefficiente tende all'infinito. Tuttavia, come è illustrato in Fig. 2,

per i valori di ϱ che danno le condizioni di ottimo nel caso precedente, il coefficiente in questione si mantiene inferiore all'unità.

Un modo alternativo di procedere è di introdurre l'isotopo radioattivo sotto forma di molecole A*B e misurare l'andamento nel tempo della concentrazione di questo composto (o, se sperimentalmente più opportuno, di AA*). Indicando con R il rapporto tra le concentrazioni istantanea ed iniziale di A*B e con R_m il valore minimo di questa grandezza, si ottiene, con procedimento analogo al precedente:
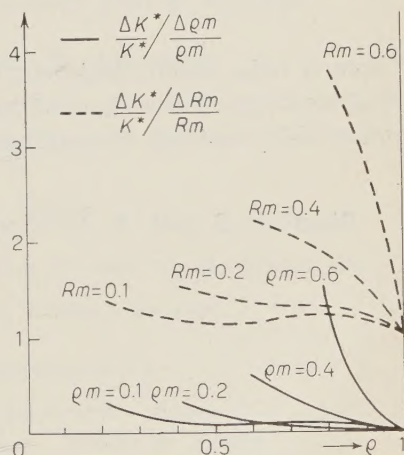

Fig. 2.

$$k^* = \frac{R_m}{t}\,\frac{2+\sqrt{E}}{c}\,\log\frac{1-R_m}{R-R_m}\,; \qquad E^* = \frac{R_m\sqrt{E}}{1-R_m}\,.$$

La misura in cui i vari errori sperimentali si ripercuotono su k^* è la stessa come nel caso precedente, ad eccezione del coefficiente di $\Delta R_m/R_m$, i cui valori sono tutti un po' maggiori. L'andamento di $(\Delta k^*/k^*)/(\Delta R_m/R_m)$ al variare di R e per diversi valori di R_m è rappresentato dalle curve tratteggiate di Fig. 2; come si vede anche in questo caso per $R_m \leqslant 0.5$ il coefficiente di ripercussione dell'errore si mantiene entro limiti accettabili anche per i valori di R che rendono minimo il coefficiente di $\Delta R/R$.

Pertanto, avendosi sempre, ovviamente, $R_m = 1 - \varrho_m$, nei casi in cui il metodo qui proposto non dà buoni risultati per la via della sintesi (cioè nei casi con $\varrho_m > 0.5$) è possibile procedere alla misura per la via della decomposizione. Nella seguente Tabella sono riportati valori approssimativi di ϱ_m ed R_m in corrispondenza ad una serie di valori delle costanti di equilibrio (assumendo $E \simeq E^*$).

| E	ϱ_m	R_m
10^{-3}	0,97	0,03
10^{-2}	0,90	0,10
10^{-1}	0,76	0,24
1	0,5	0,5
10	0,24	0,76
10^2	0,10	0,90
10^3	0,03	0,97

Come si vede, tenuto conto che il metodo proposto dà buoni risultati quando si abbia o $\varrho_m \leqslant 0.5$ o $R_m \leqslant 0.5$, esso risulta praticamente applicabile per ogni valore delle costanti di equilibrio.

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SUMMARY

The author examines the possibility of precise measurements of isotopic effects on the rate constants of bimolecular thermal equilibrium reactions of the general kind

$$A_2 + B_2 \longleftrightarrow 2\,AB$$
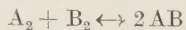
when one of the isotopes involved is radioactive. The impossibility of the measurement by means of carrier-free isotopes is evaluated through the approximated expression (1). A method is proposed whose essential feature consists of carrying out the reaction of the radioactive species

$$AA^* + B_2 \longleftrightarrow AB + A^*B \ ,$$
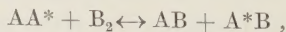
in the presence of relatively large amounts of A_2, B_2 and AB in chemical equilibrium. If the measurement is performed as suggested, the required velocity constant is given by (5). An examination of this expression from the standpoint of errors shows that the magnification factors of the experimental errors on the measured quantities are sufficiently small to allow a significant comparison between experimental and theoretical results.

The Interaction and Decay of K⁻-Mesons in Photographic Emulsion.

PART III.

(K⁻-COLLABORATION)

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Summary. — This paper presents Part III of the series on the interaction of K^--mesons in photographic emulsions contributed by the European K-collaboration. The mean free path for K^--interaction in flight $(10 \div 80)$ MeV with hydrogen and with the complex nuclei in emulsion are compared with other recent experimental data. A number of Σ-hyperons are also investigated. From favourable examples of Σ^+-p decay recorded in the emulsion the Σ^+ mass is estimated as $M_{\Sigma^+} = 2327.2 \pm 1.0 \, m_e$ and from the hydrogen capture $(K^- + H)$ events the mass difference $(M_{\Sigma^-} - M_{\Sigma^+})$ is found to be $14.6 \pm 1.1 \, m_e$: Lifetime estimates are also given for the charged Σ hyperons: a) The best estimate for the Σ^+ lifetime is obtained using only events in which the decay proton is emitted forward in the centre-of-mass system. Thus $\tau^+ = 0.82^{+0.34}_{-0.20} \cdot 10^{-10}$ s. b) A representative value for the hypothetical Σ^- lifetime as determined from 70 selected $\Sigma^{\pm}_{\pi}$ decays is $0.71^{+0.19}_{-0.12} \cdot 10^{-10}$ s. The effective lifetime appears (as well as can be ascertained) to remain of the order of the Σ^+ lifetime, even in samples containing widely different proportions of Σ^- hyperons. Although values are greater than the lifetimes previously reported in emulsion experiments, it still appears possible that a genuine anomaly may exist. The observed numbers of secondary interactions by charged particles emitted from the K^--capture stars may be accounted for on the basis of proton interactions, although some contribution from deuterons is not excluded. One definite example of a fast Σ interaction (visible energy release 129 MeV) has been found in a length of 70 cm of track registered in the emulsion. The number of single scatterings in 18 cm of Σ^- and 25 cm of Σ^+ hyperon track in emulsion (energy interval $(5 \div 100)$ MeV) are compared with that expected for Coulomb scattering by a point nucleus. Only a slight possible indication of the nuclear interaction of the Σ^--hyperon is demonstrated with the few data at present available. In K^--interactions giving $(\Sigma + \pi)$ some 45 Σ^+_p and 47 Σ^+_π decays were also examined for possible polarization of the decay with respect to the (Σ, π) plane of production.

1. – Introduction.

In the first section of the published work of the K^--Collaboration, Part I [1], the details of the K^--meson exposure and the method of scanning are given. In the ensuing portion of Part I, the results of the study on those K^--interactions in which a π-meson is emitted are presented and discussed.

In Part II [2] the data on the emission spectra of the Σ-hyperons from K^--interactions are discussed in terms of the single and multi-nucleon processes. Features of the interpretation concern the absorption in nuclear matter

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[1] K^--COLLABORATION: *Nuovo Cimento*, **13**, 690 (1959), part I.
[2] K^--COLLABORATION: *Nuovo Cimento*, **14**, 315 (1959), part II.

of the Σ-hyperons and the π-mesons released in the primary K⁻-interactions.

The present communication gives data on the interaction of K⁻-mesons of $(10 \div 80)$ MeV with the hydrogen and the complex nuclei in emulsion. Values are obtained for the mass and lifetime of the K⁻-mesons and for the charged Σ-hyperons. A preliminary report was made at the Venice-Padua Conference (1957) (³).

Certain topics relating to the properties of Σ-hyperons are also included in the present paper. These concern the Σ-interaction and scattering in nuclear emulsion, and the possible non-conservation of parity in hyperon decay. Only tentative conclusions are possible since the data are very limited.

2. - The cross-sections for K⁻-meson reactions in the interval $(10 \div 80)$ Me⁻

The frequency of interactions with emulsion nuclei for incident K-meson energies between $(10 \div 80)$ MeV is normalized to a track length of 93.5 m. The length per 10 MeV interval is shown in Fig. 1.

17 events have been interpreted as the interaction in flight of a K⁻-meson with the hydrogen nuclei in the emulsion. 11 are elastic scatters, and 6 are reaction-producing charged Σ-hyperons. Details are given in the Venice-Padua report (⁴). Events due to reactions with hydrogen nuclei leading to the production of neutral hyperons cannot be distinguished in emulsion experiments of this kind from those of similar appearance in which complex nuclei are involved. ASCOLI et al. (⁵) have combined the emulsion results from several laboratories and find, as an average over the energy range $(10 \div 100)$ MeV, between $(40 \div 50)$ mb for the elastic scattering cross-section and $(15 \div 25)$ mb for the cross-section for the reaction leading to

Fig. 1. – Observed interaction length for K⁻-mesons in nuclear emulsion.

charged Σ-hyperons, with an indication of a marked rise in the value of the latter for incident K⁻-meson energies less than 40 MeV. The number of events expected to be observed in the present experiment, on the basis of the above
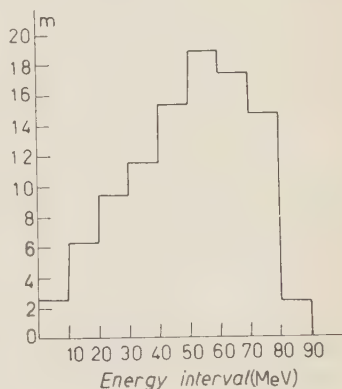
(³) *International Conference on Mesons and Recently Discovered Particles* (Padova-Venezia, 1957), Session 2, p. 5.

(⁴) Padova-Venezia Report, loc. cit., Session 2, p. 12.

(⁵) G. ASCOLI, R. D. HILL and T. S. YOON: *Nuovo Cimento*, **9**, 813 (1958).

cross-sections, is $12 \div 15$ elastic scatters and $4 \div 8$ reactions. This is in good agreement with our results.

The average mean free path in the energy interval $(20 \div 60)$ MeV for K^--meson reactions with complex nuclei is 22 ± 1 cm. In obtaining this figure, 43 out of 49 events, where the only track observed due to reaction products was of light grain density, were classified as decays in flight (see Section 3). Their inclusion as reactions would decrease the above value by 10%.
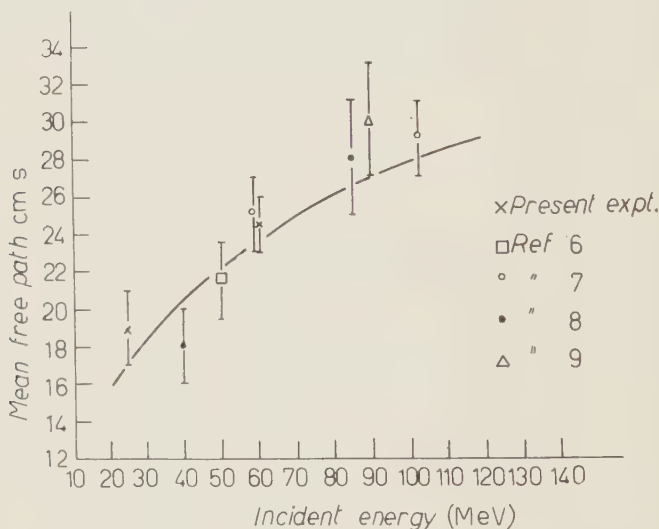

Fig. 2. – Combined experimental data for m.f.p. of K^--meson interaction with complex nuclei, normalized to a m.f.p. of 24 cm at 64 MeV. Curve fitted as described in text.

Fig. 2 shows the variation with K^--energy of the mean free path for interaction with complex nuclei. Data from a number of experiments is included ([6-9]). It is seen to be very likely that the probability of interaction decreases with a rise in incident K^--meson energy. The curve shows the expected variation taking into account both (1) the effect of a Coulomb potential of 8 MeV, and (2) the variation of the K^--meson de Broglie wavelenght λ, over the energy range considered, assuming the cross-section to be proportional to the average value of $(R_i + \lambda)^2$, where R_i is the effective radius for the emulsion nuclei.

[6] G. B. CHADWICK, S. A. DURRANI, P. B. JONES, J. W. G. WIGNALL and D. H. WILKINSON: *Phil. Mag.*, **3**, 1193 (1958).

[7] Y. EISENBERG, W. KOCH, E. LOHRMANN, M. NIKOLIC, M. SCHNEEBERGER and H. WINZELER: *Nuovo Cimento*, **9**, 745 (1958).

[8] S. NILSSON and A. FRISK: *Ark. f. Fys.*, **14**, 277 (1958).

[9] F. H. WEBB, E. L. ILOFF, F. H. FEATHERSTON, W. W. CHUPP, G. GOLDHABER and S. GOLDHABER: *Nuovo Cimento*, **8**, 899 (1958).

The curve has been normalized to the experimental average of 24 cm for the mean free path at 64 MeV. The predicted variation is in accord with the trend of the experimental points, and the mean free path appears to approach a constant value at higher energies (over 100 MeV). This could be due to the nucleus behaving as a black body to the incident K⁻-meson.

3. – Lifetime of the K⁻-meson.

The separation of decays in flight of K⁻-mesons from interactions producing the same observable features is difficult in emulsion since in general at least the range of the secondary particle is required. In the present experiment a lower limit for the lifetime was sought by classifying as decays all those events which did not show any evidence of being due to interactions. The criteria used for a decay event were (*a*) a grain density less than 1.4 times the minimum value, (*b*) the absence of any sign of nuclear excitation or interaction (*e.g.* grains that could have been due to β or γ transitions or to a recoiling nucleus). Out of 49 events where a single track was the only secondary effect, 43 were classified as being due to decays. Their distribution with energy is shown in Fig. 3. In addition 3 events interpreted as τ-decays in flight were observed.

In the emulsion used for this experiment there is considerable evidence of a strong bias against observing tracks of light grain density near the interfaces of the emulsion sheets (see Part I), so that in deducing the lifetime, neither events nor track length observed in the top and bottom

Fig. 3. - Numbers of K⁻-meson decays observed at different energies.

20 μm (unprocessed emulsion) were included. A K⁻-meson lifetime of $1.5 \cdot 10^{-8}$ s (uncorrected) was obtained with a τ-meson frequency of 6%. In Part II, the observational loss of tracks of low grain density for events (decays in flight of Σ hyperons) similar to those considered here is found to be 15% at most. Thus the lower limits for the K⁻-meson lifetime and frequency of occurrence of τ-mesons are $(1.3^{+.3}_{-.2}) \cdot 10^{-8}$ s and (6 ± 4)% respectively. This is in good agreement with previous determinations of K⁻-meson lifetime (see *e.g.* the summary of experiments in NILSSON and FRISK [10]).
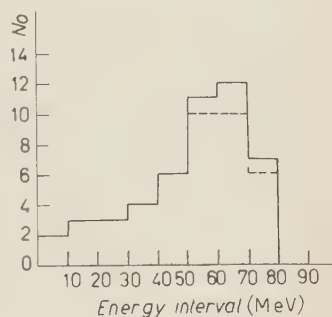
[10] S. NILSSON and A. FRISK: loc. cit., p. 293.

4. – Masses of K^--meson and Σ-hyperons.

4˙1. *Range-energy calibration of the* K_2^- *stack.* – A calibration was made utilizing the ranges of the μ^+ from various π-μ decays distributed through the stack. The emission energy of the μ-meson was calculated from recent π and μ masses to be 4.12 MeV.

Those events were selected where the μ trajectory was completed in one plate. The average dip angle in each case was less than 20°. From 83 measurements the mean μ-meson range was found to be:

$$(1) \qquad R_\mu = 599.8 \ \mu\text{m} \qquad \text{with} \qquad \sigma_m = \pm \, 3.4 \ \mu\text{m} \ .$$

The error σ_m is a variance calculated from the range straggling data of BARKAS (1955) [11]. An allowance for 5% uncertainty in the shrinkage factor (S) is included. The experimental value of σ_m as described from the spread of μ-meson ranges is $\pm 3.5 \ \mu\text{m}$.

The range R_μ includes a small parallel plate or « flat chamber » correction to allow for the non-observability of certain μ-meson ranges with inclined trajectory.

The range corresponding to 4.12 MeV μ-meson energy is estimated for the Barkas standard G-5 emulsion to be

$$(2) \qquad\qquad R_\mu(s) = 602.5 \ \mu\text{m} \ .$$

Comparison of this result with (4.1) above establishes a calibration point for the present stack.

4˙2. *Range of protons from* Σ_p^+ *decay at rest.* – Six protons were selected for range measurement where the tracks were all contained in one plate, thus eliminating the uncertainty in range associated with the crossing from one plate to another. For each event the processing shrinkage factor (S) of the emulsion was determined. The method employed utilized data from the 8.776 MeV α tracks originating from ThC' stars in the same plate. The mean shrinkage (S) for the stack is about 2.06, but individual plates showed measurable variations $\sim 5\%$ of the mean value.

The non-linear distortion was determined in the vicinity of each proton track by the vector method of Cosyns and Vanderhaege [12]. Values of the

[11] W. H. BARKAS, F. M. SMITH and W. BIRNBAUM: *Phys. Rev.*, **98**, 605 (1955).

[12] M. G. E. COSYNS and G. VANDERHAEGHE: *Bull. du Centre Phys. Nucleare Université Libre de Bruxelles*, no. 15 (1951).

2nd order distortion vector K_2 ranged from 20 μm to 70 μm. Correction to range for first and second order distortion were applied assuming $K_2 = -2K_1$ (APOSTOLAKIS and MAJOR)[13]. Since all tracks were of small dip (15° or less) the actual range correction did not exceed 11 μm in any one case.

The corrected proton ranges are: 1637, 1639, 1655, 1658, 1685 and 1711 μm.

The « flat chamber » correction necessary to give an unbiassed mean from the selection of tracks stopping in one plate is 0.1 μm which when added gives for the mean range of protons stopping in the one plate:

$$(3) \qquad\qquad R_p^1 = (1663 \pm 10) \text{ μm} .$$

The uncertainty in the mean σ_{m-1} (above) as calculated from straggling, shrinkage errors, etc., is consistent with the observed spread of ranges for which $\sigma_m = 10.0$ μm.

The mean range of the proton from $R\Sigma_p^+$ decay corrected to the standard (BARKAS [14]) emulsion is:

$$(4) \qquad\qquad R_p(s) = (1670 \pm 14) \text{ μm} .$$

The standard error quoted includes the uncertainties in the stopping power of the emulsion.

4˙3. *Range measurement of* Σ^+ *and* Σ^- *hyperons from* K⁻-H *captures at rest.* –
As is well known the capture of a K⁻-meson at rest by an unbound proton is characterized by a star consisting of two collinear prongs at 180° one being of a Σ-hyperon and the other of a π-meson. A complete list of hydrogen capture events is given in Table I.

Since π and Σ are produced in a two-body process the momenta of Σ^+ and Σ^- have unique values. This is manifested in characteristic ranges for Σ^+ and Σ^--hyperons.

Particular care was taken in this experiment to measure these ranges as accurately as possible. The projected track lengths were obtained from repeated stage micrometer measurements and a number of depth readings was made with a 95 × objective at points on the trajectory. The processed thickness of the emulsion was measured at the same time.

Careful shrinkage and distortion measurements were made in each plate concerned in the manner described in Section 3 above.

Table I gives details of the Σ^- and Σ^+ ranges from the K⁻-H captures. The range corrected for 1st and 2nd order distortion is given in column 7. The estimated total error ε for each event includes $\pm 1.6\%$ range straggling

(13) A. J. APOSTOLAKIS and J. V. MAJOR: *Brit. Journ. Applied Phys.*, **8**, 9 (1957).
(14) W. H. BARKAS: *Nuovo Cimento*: **8**, 201 (1958).

TABLE I. – *Range measurements of Σ^+ and Σ^- hyperons from K^--H captures.*

Event No.	No. of plates	Ending	Dip Angle	Measd. Range (μm)	Distortion (corrn.) (μm)	Final [a] Range (μm)	Total Error (ε) (μm)
			Σ^+ hyperons				
5151	1	$R\Sigma_p$	18°	834.8	— 6.8	828.0	±14
9/15	1	$R\Sigma_p$	13°	812.0	+ 4.0	816.0	±13
DP6 (**)	2	$R\Sigma_p$	40°	814.7	0.0	814.7	±21
449	2	$R\Sigma_\pi$	43°	816	(*)	(816)	(±23)
			Σ^- hyperons				
96D1	1	Σ_p [1]	25°	718.1	— 2.6	715.5	±13
1991 (**)	1	Σ_p	45°	673.9	+ 4.5	678.4	±20
4991 (**)	1	$\Sigma_{\sigma 3}$ [3]	45°	692.5	— 0.3	692.2	±11
1891 (**)	1	$\Sigma_{\sigma 1}$ [2]	0°	698.9	0.0	698.9	±11
Mi_{67}	1	Σ_p	35°	684	(*)	(684)	(±16)
Pd_{125}	1	$\Sigma_{\sigma 1}$ [2]	1°	697	(*)	(697)	(±11)
3293	2	Σ_p	61°	687.3	+ 2.2	689.5	±28
3/35	2	$\Sigma_{\sigma 1}$ [2]	6°	704	0.0	704	±11
Pd_{61}	2	Σ_p	29°	658	(*)	(658)	(±13)
Pd_{377}	2	Σ_p	45°	690	(*)	(690)	(±20)

[a] Corrected range in the present stack.
(*) Not measured for these events.
(**) Hydrogen captures additional to those found in the line scan.
[1] Indicates Σ track with ϱ'-ending.
[2] Indicates Σ^- star with 1 prong.
[3] Indicates Σ^- star with 3 prongs.
Also observed were five Σ-decays in flight from K^--H capture at rest: one $F\Sigma_p^+$, two $F\Sigma_\pi^+$ and two $F\Sigma_\pi^\pm$.

and the effect of the 5% uncertainty assumed in the shrinkage factor determined for each plate.

A statistical weight of $100/\varepsilon^2$ has been assigned to each range measurement according to its accuracy.

The weighted mean range for the four Σ^+ is found to be:

(5) $$R(\Sigma^+) = (820 \pm 8) \ \mu m$$

and for the 10 Σ^-:

(6) $$R(\Sigma^-) = (693 \pm 4) \ \mu m \ .$$

The ranges of the hyperons normalized to the Barkas standardized emulsion become:

(7)   for Σ^+: $R_s(\Sigma^+) = (824 \pm 9)$ μm

(8)   and for Σ^-: $R_s(\Sigma^-) = (696 \pm 6)$ μm .

The errors quoted incorporate the uncertainties in the range energy calibration of the present stack.

The difference in ranges is:

(9) $$\Delta R_s = R_s(\Sigma^+) - R_s(\Sigma^-) = (128 \pm 9) \text{ μm} .$$

The value of ΔR_s is almost independent of the range energy calibration employed.

4˙4. *Masses*. – The masses of K^-, Σ^- and Σ^+ may be found from the data for the reactions studied in (4.2) and (4.3). The values for p, π^0 and $\pi^\pm$ are given by CROWE ([15]) as: 1836.12, 264.27 and 273.27 times electron mass respectively.

a) **Mass the of Σ^+-hyperon.** This may be found from $\Sigma^+ \to p + \pi^0$ alone. Using the data for the 6 events in one plate above (4), the energy of the decay proton is found to be $E_p = (18.82 \pm 0.05)$ MeV. The result for the Σ^+ mass is:

(10) $$\begin{cases} M_{\Sigma^+} = (1\,189.2 \pm 0.5) \text{ MeV} , \\ = (2\,327.2 \pm 1.0) \text{ m}_e . \end{cases}$$

b) **Mass of the K-meson.** The energy of the Σ^+ corresponding to the $R_s(\Sigma^+)$ found from the K^--H capture: $K^- + p \to \Sigma^+ + \pi^-$ is equal to (13.74 ± 0.08) MeV. Using this result and the Σ^- mass above, the K^- mass is found to be

(11) $$\begin{cases} M_{K^-} = (493.6 \pm 0.7) \text{ MeV} , \\ = (966.1 \pm 1\,5) \text{ m}_e , \end{cases}$$

or,

c) **The mass difference $M_{\Sigma^-} - M_{\Sigma^+}$.** A comparison of the ranges of Σ^- and Σ^+ from the K-H captures provides an accurate means of determining the mass difference $\Delta M = (M_{\Sigma^-} - M_{\Sigma^+})$. This method has the ad-

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([15]) K. M. CROWE: *Nuovo Cimento*, **5**, 541 (1957).

vantage that the result is relatively insensitive to the value assumed for the K^- mass. It is also less dependent upon the accuracy of the range-energy calibration of the stack since to a first approximation the mass difference is proportional to $\Delta R = R^+ - R^-$ only.

i) **From events in one plate only.** Using only the data for tracks contained in one plate yields $\Delta R = (126 \pm 11)$ μm and the resulting mass difference is

(12) $$\Delta M_1 = (14.4 \pm 1.3) \, \mathrm{m_e} \, .$$

ii) **For events in two plates.** It is necessary here to take into account how seriously the measured ranges will be affected through the possible loss of sensitive emulsion from surface abrasion etc., resulting from the final cleaning of the plates after processing. A thin silver obscuration was removed with a fine abrasive as was previously done in the G-stack. As far as it was possible to measure it in the latter the thickness of the sensitive layer removed was found to be of the order of 1 or 2 μm (uncertainty ± 2 μm). Hence although the amount is not well ascertained it is believed likely to be small in the present stack. As ΔM is proportional to a range difference ΔR, some cancellation of the resultant error in ΔM will result.

It is calculated that the removal of 1 μm of sensitive emulsion would change the mean ΔR (as weighted in Table I) by 4 μm. If all events are given equal weight the figure is reduced to 2 μm. Since the available data is limited and these errors are appreciably smaller than the uncertainty in ΔR in i) above, (*i.e.* ± 11 μm) it is considered legitimate to incorporate the crossing events with those which are contained in one plate only.

iii) **Thus for all events**, using the weighted mean range for the 4 Σ^+ and 10 Σ^- yields $\Delta R = (128 \pm 9)$ μm and

(13) $$(M_{\Sigma^-} - M_{\Sigma^+}) = (14.6 \pm 1.1) \, \mathrm{m_e} \, .$$

This differs only slightly from the value found for Σ events in one plate alone [(12) above] and is the value adopted for this paper.

5. – Lifetime of the Σ-hyperons.

A report is given of an evaluation of the mean lifetimes for the charged hyperons produced following capture of K^--mesons at rest in nuclear emulsion. The hyperons studied are associated with the 3035 K^--stars examined by the European Collaboration.

5˙1. *Method.* – There are two methods available for the determination of the lifetime of unstable particles.

a) The first method uses data for decays in flight only ([16]). The best estimate of the mean life is given by the solution $\hat{\tau}$, of the maximum likelihood equation:

$$(1) \qquad f(\tau) = \sum_{i=1}^{n} \left[\frac{t_i}{\tau} - 1 + \frac{T_i}{\tau} \cdot \frac{U_i}{1 - U_i} \right],$$

where: $U_i = \exp\left[-T_i/\tau\right]$ and

   t_i is the time of flight of the ith particle in the emulsion before decay; and

   T_i is the « Moderation time » for the ith particle, the time for which the ith particle would have travelled in the emulsion had it not decayed.

Following Bartlett's method ([16,17]) the estimate $\hat{\tau}$ and its standard error are obtained by plotting against the quantity $1/\tau$ the function:

$$(2) \qquad s(\tau) = f(\tau) \left| \sum_{i=1}^{n} \left\{ 1 - \left(\frac{T_i}{\tau}\right)^2 \cdot \frac{U_i}{(1 - U_i)^2} \right\} \right|^{-\frac{1}{2}},$$

where $f(\tau)$ is given by equation (1).

The expected value of $s(\tau)$ is zero; its variance is 1.

b) In the second, the « rest+flight » method, both decays in flight and those at rest are considered.

The estimate of the mean life is given by

$$(3) \qquad \hat{\tau} = \frac{1}{n} \left[\sum_{r=1}^{n} t_r + \sum_{s=n+1}^{N} T_s \right],$$

where: n is the number of decays in flight;

   $N - n$ is the number of decays at rest;

   t_r is the time of flight of the ith particle which decayed in flight;

   T_s is the time of flight of the ith particle which decayed at rest.

([16]) M. S. BARTLETT: *Phil. Mag.*, **44**, 249 (1953).
([17]) C. FRANZINETTI and G. MORPURGO: *Suppl. Nuovo Cimento*, **6**, 577 (1957).

Again following BARTLETT, the variance of $\hat{\tau}$ is deduced from the function

(4) $$s(\tau) = \frac{n[\hat{\tau}/\tau - 1]}{[n + N - \sum_{q=1}^{N} \exp[-T_q/\tau]]^{\frac{1}{2}}},$$

which has the expected value 0 and variance 1.

In this experiment it has not in general been possible to determine the sign of those hyperons which decay into a charged π meson and a neutron. If we assume hypothetically that all the Σ_π^- decays in flight are produced by a homogeneous group of particles with a unique mean life, we can estimate the mean life by applying the first method. However, bubble chamber results indicate that this assumption is almost certainly false. We can therefore base only indirect arguments on the estimate that we obtain from this method.

If we attempt to apply the second method we should have the additional complication of a serious bias; for a considerable though poorly defined fraction of the $\Sigma^+ \to \pi$ decays at rest will escape observation; and about $\frac{2}{3}$ of the stopped Σ^- particles yield no recognizable star. In these circumstances a mean life estimate derived by the second method is of little significance.

5`2. *Selection of the data.* – The numbers of Σ decays observed from the 3 035 stars produced by K^- at rest are

    i) Decay in flight to proton: $R\Sigma_p^+$: 61

    ii) Decay at rest to proton: $F\Sigma_p^+$: 39

    iii) Decay in flight to π-meson: $F\Sigma_\pi^+$: 81 .

We give below some comments on the identification and measurements of the particles in the three groups.

a) $R\Sigma_p^+$ – All decays in this group should be detected because of the characteristic range of the secondary proton. The contamination due to Σ^- absorption yielding a secondary track of range about 1670 µm is considered to be small (see Section 2-*b*) of Part I).

b) $F\Sigma_p^+$ – Every baryon track in which there is a deflection associated with an ionization change has been considered as a possible $F\Sigma_p^+$. The angles, the range of the secondary particle and the velocity of the primary particle have been measured. The velocities were measured either by ionization (gap length, blob density, etc.) or by scattering. Kinematic analyses of the assumed decays have been made. It has thus been possible to eliminate most of the scatterings and interactions of protons and hyperons but, as already has been

noted in Part II ([18]), a proton which is emitted backwards in the CMS may give rise in the laboratory system to an event in which the angle of deflection is small and the velocities of the primary and secondary tracks are similar. Such events would not be recognized as decays. It is particularly difficult to detect changes in either ionization or direction when the dip of the primary particle is large and the errors in measurement of both ionization and scattering are substantial. Events for which the dip of the primary particle is $> 45°$ have therefore been excluded from the data on which the mean life has been estimated.

c) $F\Sigma_\pi^\pm$ – The end points of all baryon tracks which stopped in flight have been carefully searched for secondary π-mesons.

Because of the finite dimensions of the stack, and because of the difficulty of following a minimum ionization track from one plate to the next, it has only been possible to follow one secondary π-meson to the end of its range. In the remaining events, neither the signs nor the energies of the secondary π-mesons are known; it has been possible only to measure the energies of the hyperons.

It has however been possible in some cases to determine the sign of a π-meson associated with a Σ-particle, i.e. where a π-meson is produced in the same K-star. In such cases we assume that the sign of the Σ-particle is opposite to that of the π-meson. This permits us to make mean life estimates for subgroups of Σ-particles of known sign. These subgroups are unfortunately very small.

Finally, where there is a fast hyperon ($E > 60$ MeV) associated with a fast proton ($E > 60$ MeV) it appears reasonable to assume the reaction

$$(5) \qquad\qquad K^- + p + n \rightarrow \Sigma^- + p$$
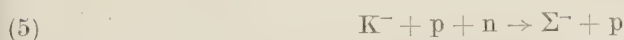
and thus to form another subgroup which can be used for an estimate of the mean life of the Σ^- particle.

Here again, as in class b), because ionization measurements on steeply dipping tracks are subject to large errors, we have excluded all events in which the dip of the primary particle is $\sim 45°$. This restriction is also necessary because when the primary particle dips steeply it is difficult to distinguish between decays in flight and decays at rest.

The number of hyperons with dips $< 45°$ is

$$F\Sigma_\pi^\pm: \quad 70\ (*) \qquad F\Sigma_p^+: \quad 31\ .$$

([18]) K⁻-COLLABORATION: *Part* II (loc. cit.), Sect. **2**.
(*) The total number if disappearances in flight are included as unrecognized $F\Sigma_\pi^\mp$; is increased to 85. (See Sect. 4·3)).

5'3. *The mean life of* Σ_p^+ .

a) From the events Σ_p^+

$$\boxed{\tau^+ = \left(0.68^{+0.41}_{-0.19}\right) \cdot 10^{-10} \text{ s}} \ .$$

(T1)

If all the events had been considered, regardless of dip angle, we should have found

(T1a) $$\tau^+ = \left(0.94^{+0.65}_{-0.27}\right) \cdot 10^{-10} \text{ s} \ .$$

b) From the events $(R+F)\Sigma_p^+$.

In order to eliminate the bias due to the difficulty in detecting some of the $F\Sigma_p^+$ events in which the proton is emitted backward in the CMS we have derived the mean life using only those events in which the proton is emitted forward in the CMS.

The best estimate is

(T2) $$\boxed{\tau^+ = \left(0.82^{+0.34}_{-0.20}\right) \cdot 10^{-10} \text{ s}} \ .$$

The above results for Σ^+ hyperons are summarized in Table II. They agree with results obtained in other experiments with nuclear emulsions and with bubble chambers lifetimes reported at the CERN Conference ([19]). These results are also given in Table II.

TABLE II. – *Estimates of the mean life of the* $\Sigma^+ \to p$ *particles.*

Data	No. of events	Sum of $T_i \cdot 10^{10}$ s	Mean life $\cdot 10^{10}$ s
$F\Sigma_p^+$	31	37.86	$0.68^{+0.41}_{-0.19}$
$(R+F)\Sigma_p^+$	R: 22 (*) F: 19	10.21 28.99	$0.82^{+0.34}_{-0.20}$
Bubble Chamber ([19])	—	—	$0.75^{+0.10}_{-0.09}$
Other emulsion results ([19])	—	—	$0.93^{+0.10}_{-0.08}$

(*) Selected data. Only events emitted into the forward hemisphere C.M.S. considered, dip angle restricted to $< 45°$. (Some correction applied to time of flight of $R\Sigma_p^+$).

([19]) *CERN conference on High Energy Nuclear Physics* (Geneva, 1958).

5·4. *Lifetime estimates derived from the events* $F\Sigma_\pi^\pm$.

a) If the 70 $F\Sigma_\pi^\pm$ events (with primary dip angle $< 45°$) are assumed to form a homogeneous group the estimate of the hypothetical unique mean life is

(T3)
$$\tau^\pm = \left(0.71^{+0.19}_{-0.12}\right) \cdot 10^{-10} \text{ s}$$

.

It should be noted that for these events the evaluation of t_i and T_i is generally derived from ionization measurements; the lower the velocity of the particle the greater the error in this measurement; and when the residual range is small there is a danger of classifying as a decay in flight an event which is really a decay at rest. We have therefore made three re-estimates of $\tau^\pm$ after applying 3 different cut-offs, *i.e.* we have replaced T_i by $T_i - T_c$, where T_c for the three estimates is $0.2 \cdot 10^{-10}$ s, $0.3 \cdot 10^{-10}$ s and $0.4 \cdot 10^{-10}$s. We have excluded events which yield negative values of $T_i - T_c$, *i.e.* for the three cases we have neglected particles with residual range 572, 987 and 1556 µm. The estimates thus obtained are given in Table III. The best estimate of the mean life tends to increase as T_c is increased.

TABLE III. – *Effect of cut off on estimates of the* « *mean life* » *of* $\Sigma_\pi^\pm$
(from $F\Sigma_\pi^\pm$ *events* $< 45°$ *dip).*

Cut off time ($\cdot 10^{10}$ s)	Cut off range (µm)	No. events	Sum of $T_i \cdot 10^{10}$ s	Mean life $\cdot 10^{10}$ s
0	0	70	120.60	$0.71^{+0.19}_{-0.12}$
0.2	572	54	103.48	$0.76^{+0.22}_{-0.14}$
0.3	987	50	97.63	$0.80^{+0.24}_{-0.15}$
0.4	1556	48	91.59	$0.83^{+0.27}_{-0.16}$

However it may be seen that the difference between the extreme values is not statistically significant; this indicates that no serious bias is introduced by the difficulty of distinguishing between decays at rest and decays in flight of low velocity particles.

b) We have already emphasized that the estimates recorded in Table III are based on the assumption that all the $F\Sigma_\pi^\pm$ particles have the same mean life. The estimates agree with those obtained for the Σ^+ particle alone. This suggests that either:

i) there is little difference between the Σ^- mean life and that of the Σ^+ particle, or

ii) that our mixture of $F\Sigma^\pm_\pi$ is very rich in Σ^+ particles.

The first hypothesis conflicts with bubble chamber results (CERN Conference, loc. cit.) which indicates that the mean life for the Σ^- is almost double that for the Σ^+-hyperon.

In view of the above results we have made mean life estimates on various subgroups which are known to contain different proportions of the two constituents Σ^+ and Σ^-. The results are given in Table IV.

TABLE IV. – *Estimates of the « mean life » of $\Sigma^\pm_\pi$ for groups of events containing different proportions of Σ^-.*

Type of event	Estimated % age of Σ^-_π	Observed no. of decays	Sum of $T_i \cdot 10^{10}$ s	Mean life (τ) $\cdot 10^{10}$ s
$F\Sigma^\pm_\pi$ with $\pi^\pm$ at K$^-$-star [a]	$31 \pm 6\%$ [f]	31	19.0	$0.97^{+0.62}_{-0.45}$ [*]
$F\Sigma^\pm_\pi$ (Total) [b]	$47 \pm 5\%$ [f]	70	120.6	$0.71^{+0.19}_{-0.12}$
$F\Sigma^\pm_\pi > 60$ MeV [c]	$49 \pm 5\%$ [g]	32	101.3	$0.64^{+0.16}_{-0.11}$
$F\Sigma^\pm_\pi$ without $\pi^\pm$ at K$^-$-star [d]	$68 \pm 6\%$ [f]	39	101.6	$0.69^{+0.19}_{-0.13}$ [*]
Identified Σ^--decays [e]	100% [e]	12	18.1	$0.71^{+0.74}_{-0.24}$

(a) A charged π-meson accompanies the $\Sigma^\pm$ hyperon from the K$^-$-star.
(b) For the whole group. Mean life estimate repeated from Table II.
(c) Energetic Σ-hyperons, with emission energy over 60 MeV.
(d) The Σ-hyperons originate from K$^-$-stars in which a charged π-meson is not emitted.
(e) The charge of the hyperon inferred, either from presence of (1) π^+-meson from K$^-$-star, or (2) fast proton and hyperon both emitted with more than 60 MeV.
(f) From Table II corrected numbers of hyperons in Part II.
(g) From Table IV corrected numbers of hyperons over 60 MeV.
(*) It should be noted that the data used for the composite lifetimes in Table III are not all independent of each other. The lifetimes however from (a) 31% Σ^- and (d) 68% Σ^- are independent, since these are based on completely separate samples of events.

The percentages of Σ^- in the different subgroups (given in column 2) are those derived in Part II.

The different estimates of the mean life are given in column 5. There is no sign of an increase in $\tau^\pm$ as the fraction of Σ^- is increased, although the mean life of Σ^- is now considered to be almost double that of the Σ^+.

5˙5. *Loss in the detection of* $F\Sigma_\pi$ *decays - Effect on lifetime estimates.* − Observation loss of decay may arise; *e.g.* if the depth within the pellicle at which the Σ decays to π-meson is close to the upper or the lower surface of emulsion, the lightly ionizing secondary track may well not be located. Moreover it is evident from the known scanning loss for lightly ionizing tracks in the present stack, that a percentage of the secondaries may well have been missed, even when the point of decay is favourably placed in the emulsion. These loss factors are discussed in relation to the observation of a number of apparent disappearances in flight, or « stops » of baryon tracks in Section 6˙2 of this paper.

The conclusion is reached that the substantial majority of the « stops » probably represents $F\Sigma_\pi$ decays with non-detected secondary.

On the latter hypothesis the « Σ-lifetime » calculated using only data from 15 « stop » events with dip angle $< 45°$ is:

(T4) $$\tau_{\mathrm{stops}} = (1.41^{+0.33}_{-0.36}) \cdot 10^{-10} \text{ s .}$$

For the purpose of examining its effects, if the « stops » are included with the $F\Sigma_\pi^\pm$ data, the lifetime (T3) is increased and becomes (*)

(T5) $$\tau^\pm \approx (0.90^{+0.20}_{-0.13}) \cdot 10^{-10} \text{ s .}$$

Similarly the lifetime estimates $a)$, $c)$ and $d)$ in Table III would also be somewhat increased.

5˙6. *Indirect estimate of* Σ^+ *lifetime.* − If the Σ^+ lifetime can be regarded as established, then the Σ^- lifetime may be estimated indirectly with the « rest + flight » time method as follows:

The number (n^-) of $F\Sigma_\pi^-$ and the corresponding summation moderation time for these hyperons decays is found by the process of subtracting out similar estimates for the $F\Sigma_\pi^-$ from the combined data for $F\Sigma_\pi^\pm$. Assuming that the Σ^+ modes have the same decay constant then the number (n^+) of $F\Sigma_\pi^+$, is equal to the number of $F\Sigma_p^-$ which is readily observed, and for which the data is available.

The correct numbers of Σ^- coming to rest has been estimated in Part II.

In Table V the moderation and flight time data necessary for the estimate of Σ^- has been summarized using the corrected hyperon numbers from Part II.

(*) The result $(T5)$ given can only be regarded as approximate, in view of the uncertainties concerning the loss of $F\Sigma_\pi$ events.

TABLE V. – *Data used in the indirect estimate of the* Σ^-*-lifetime.*

| Type of event | Number of events | | Total flight time (*) $\Sigma t_r^{\pm} \cdot 10^{10}$ s | Total moderation time (*) $\Sigma T_s \cdot 10^{10}$ s |
	Observed	Corrected		
(a) $F\Sigma_\pi^{\pm}$	70 (**)	133 (*•*)	56.2	229.2
(b) $R\Sigma_\pi^+$	47 (**)	74 (*•*)	—	24.7
(c) $R\Sigma^-$	63	189 (*•*)	—	97.5

(*) Based on data for observed events and adjusted to corrected numbers of hyperons.
(**) Dip angle $0° \div 45°$.
(*•*) Dip angle $0° \div 90°$.

The number of $F\Sigma_\pi^+$ decays (n^+) is first found using Eq. (3) with $\tau^- = 0.8 \cdot 10^{-10}$ s, and, substituting from Table V the values for the mean time of flight obtained from the $F\Sigma_p^+$ (*i.e.* $0.28 \cdot 10^{-10}$), we may infer that $n^+ = 48$. Hence we deduce that the number of $F\Sigma_\pi^-$ decays $(n^-) = 133 - 48 = 85$.

Again using the same method and substituting $\Sigma t_r^- = \Sigma t_r^{\pm} - \Sigma t_r^+$ with $n^- = 85$, we obtain

$$\tau^- = 1.65 \cdot 10^{-10} \text{ s } (*). \tag{16}$$

5˙7. *Discussion. The $\Sigma^{\pm}$ lifetime anomaly.* – The lifetime estimates (T1) and (T2) here reported for the Σ^+-hyperon obtained from the data for Σ_p decays are in quite good agreement with the best available estimates [19,24].

It has been generally found however, that an anomalously short effective lifetime for the composite $F\Sigma_\pi^{\pm}$ decay events has resulted from the analysis of events recorded in nuclear emulsions.

Excluding the present work the values reported at the CERN Conference (1958) [19] are from 0.31 to $0.51 \cdot 10^{-10}$ s. The present estimates (with cut-off residual range) (see Table III) indicate values of $\Sigma^{\pm}$ lifetime in the vicinity of $0.80^{+0.21}_{-0.15} \cdot 10^{-10}$ s. If allowance for lost events is made (Section 5˙5) the value of the $\Sigma^{\pm}$ lifetime should perhaps be increased nearer to $0.90 \cdot 10^{-10}$ s.

Since the latter figures are still only about equal to the Σ^+-lifetime, it seems that some anomaly may be present, especially as there appears to be no change in effective lifetime (as well as can be discerned) as the proportion of Σ^- in the sample is increased (Table IV). On the other hand this is in conflict with the indirect estimate (5.6) using (rest-flight) data which indicates a Σ^--lifetime of the right order.

(*) Subject to the uncertainties in the Σ^+ branching ratio and the number of Σ^- stars at rest.

Hence, although there are some features of the present results which would seem to confirm the existence of an anomalous behaviour in the $F\Sigma^{\pm}_{\mp}$ decay processes (*e.g.* the existence of the short lived Σ^- which has been suggested), the case is not quite so striking as the previous emulsion work would seem to indicate.

6. – Interactions of Σ-hyperons and of protons.

In the course of following out the prongs from the K⁻ stars a number of interactions of unidentified baryons was recorded. A total of 28 particles produced secondary stars of one or more branches. A further 18 particles disappeared in flight with no visible star or, in the case of two events, showing only a short electron track in evidence.

6'1. *Interactions with formation of visible stars.* – In general except in the cases where there is exothermic energy release in the secondary star it is not usually possible to distinguish interactions due to fast protons from those due to Σ-hyperons.

These will be referred to as « FBy interactions » in what follows. Since the ratio of protons to charged Σ with energy > 30 MeV from K⁻-stars is known to be ~ 5 to 1 it can be anticipated that many of the FBy stars observed are likely to originate from the interactions of protons. In order to reach more definite conclusions a study of the length observed in emulsion for Σ tracks and for proton tracks is therefore necessary.

Fig. 4. – Total track length of Σ-hyperons examined in nuclear emulsion.

Fig. 4 shows the total length of Σ track in the emulsion available for examination (dip angle $< 50°$). Data for the total length of proton track is also summarized in Table VI together with the observed numbers of interactions and « stops » associated with the baryon prongs from π and non-π emitting K⁻ stars.
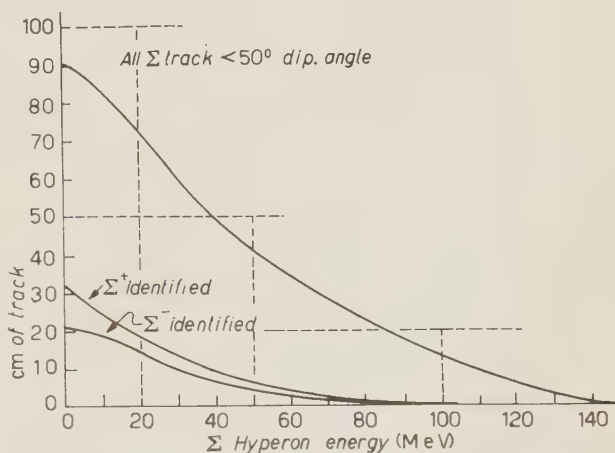
TABLE VI. – *Observed numbers of interactions in flight of baryons in relation to the total track length of protons associated with each type of* K⁻ *star.*

Total length of track (dip angle less than 50°) for stable prongs:	K stars at rest without charged Σ or H. F.	
	Stable prongs only	$\pi +$ one or more stable prongs
(1) of all energies	12.0 m	1.5 m
(2) for $E > 20$ MeV	9.0 m	1.1 m
No. of stars produced by interaction in flight of a baryon	18	5
No. of disappearances in flight or « stops » observed (*)	14	1

(*) Also 5 interactions and 3 stops observed where the primary trajectory had inclination of over 50°. One « stop » of a proton in association with an identified hyperon from a K⁻-star was recorded.

The primaries of the FBy interactions listed in Table VI may be Σ^+, Σ^- or protons since stars with identified Σ or HF have been excluded. Possibly a small contribution due to deuterons may also be present. The greatest part of the proton track is evidently associated with K⁻ stars of the category « stable prongs only » and here it may be seen the number of interactions observed is also greatest.

The interaction cross-section for fast protons in emulsion is known to be about geometric at 240 MeV (LADU [20]) and also at 130 MeV (LEES *et al.* [21]). The interaction length is about 35 cm. It is unlikely to change much within the range (130 ÷ 240) MeV.

Precise information is not available at lower energies. However the interaction length for protons may be found from the data now available for the cross-sections for neutron capture in various elements, it allowance is made for the Coulomb repulsion of the protons from the nuclei. Using the results obtained by VOSS and WILSON [22] for Cu, Cd and C absorbers with a weighting appropriate to the proportion of light and heavy nuclei in the emulsion, it appears that the interaction length for protons equal to say λ_p at 130 MeV, should decrease to a minimum of $0.79\lambda_p$ in the vicinity of 55 MeV and rise

[20] M. LADU: *Nuovo Cimento*, **10**, 855 (1953).
[21] C. F. LEES, G. C. MORRISON, H. MUIRHEAD and W. G. V. ROSSER: *Phil. Mag.*, **44**, 304 (1953).
[22] R. VOSS and R. WILSON: *Proc. Roy. Soc.*, A **236**, 41 (1955).

again to $1.0\lambda_p$ at 20 MeV. The results of calculations are shown in Table VII for the expected number of interactions and disappearances in flight or « stops » created by protons $(20 \div 170)$ MeV from K⁻ stars on the basis of the Lees *et al.* data for star formation and prong multiplicity at 130 MeV.

TABLE VII. – *Calculated and observed numbers of interactions of prongs from* K⁻ *stars.*

	Classification of interaction stars			
	0-prong	1-prong	2 or more prongs	Remarks
(A) *Primary energy* $(20 \div 170)$ MeV Expected No. of proton interactions (^{a})	1	Total 22		Calculated as described in text, using results of LEES *et al.* (^{21}) (^{b})
Observed No. of interactions of prongs from K⁻ stars	13	8	9	Primary trajectory with dip angle $< 50°$
(B) *Primary energy* $(5 \div 20)$ MeV Observed No. of interactions	1	6	0	Number apart from elastic scattering events.

(a) Computed total track length in emulsion with dip angle $< 50°$.
(b) Data for 60 m of proton track at 130 MeV.

A cut-off of 50° has also been applied to the permissible dip angle to allow the estimate to be compared with the experimental data. Thus the number of stars (with one or more prongs) created by the protons should be about 22. The number actually observed was 17. Thus whilst our knowledge is not precise, it is evident that the interaction of protons will explain the origin of the majority of the one or more branch secondary stars observed.

Further information is provided from the study of the energy release in the interaction stars. For this purpose the visible energy release (E_t) may be estimated as: $E_t = E_{kin} + 8n$ (MeV), *i.e.* as the sum kinetic energy of the n prongs plus an average binding energy of about 8 MeV for each prong. Only stars which show a definite exothermic energy release (E_t) exceeding the kinetic energy of the interacting primary E_s may be safely identified as $F\Sigma$ interactions in flight. Further it is necessary that the track of the primary particle should exceed about 1 mm in length, otherwise the identification and the energy estimate E_s, both tend to be uncertain.

Excluding two interactions formed at less than 1 mm from the K⁻-capture

star, only the interaction described in Table VIII may be claimed to be definitely exothermic and ascribed to $F\Sigma^{\pm}$ interaction with an emulsion nucleus. Thus one $F\Sigma^{\mp}$ interaction has been detected in a length of about 70 cm of Σ-track examined (using data of Fig. 4).

TABLE VIII. – *Exothermic* $V\Sigma^{\pm}$ *interaction* (Event no. 7253).

(a) Data on interaction star

| Energy of Σ at interaction (E_s) | Interaction star formed by Σ-hyperon | | | Identification of primary |
	No. of prongs	Total energy release (E_T)	Prong energies (MeV)	
52 MeV	6	129 MeV	33, 25, 9, 5, 5, 4	probable Σ^-

(b) Data on primary hyperon

Description of K star	Dip angle	Emission energy	Range	Scattering measurements
$\Sigma + p$ (70 MeV) + 2p	5°	99 MeV	20.9 mm	Mass from scattering = 2400 ± 300 m_e

6'2. Disappearances in flight or «stops». – A total of 20 apparent disappearances in flight of baryon tracks were recorded. After re-examination of these events one was revealed as a $V\Sigma^{\pm}_{\pi}$ in which the decay π had not originally been detected.

The remainder of 19 events provisionally classified as «stops» are shown in Table VI. Of these there are 16 in which the primary may be either Σ or p and are within the geometry of the experiment (*i.e.*, dip angle is less than 50°). It will be seen from Table VII that protons will contribute comparatively few (~ 1) zero-prong interactions and will not account for the number observed. In view of the fact that the scanning loss for lightly ionizing π-tracks was considerable in this stack ([23]), it may be postulated that the remainder of the apparent «stops» are due to the non-detection of the secondaries from $V\Sigma^-_{\pi}$ decays. This should especially apply in conditions where the π range in the pellicule is short, *i.e.* when the decay occurs near either surface of the emulsion. The distribution of stops in depth in fact shows that few were recorded within the central 200 μm of the emulsion (Fig. 5). Further

[23] Part I, (loc. cit.), p. 701.

the depth distribution of observed $V\Sigma_\pi^\mp$ events suggests that there has been a loss within 150 μm of surface or glass. On the π-loss hypothesis when the two distributions are added a uniform depth distribution of track endings
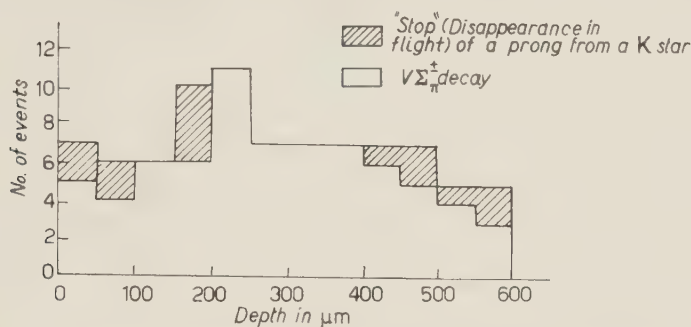

Fig. 5. – The observed distribution in location of $V\Sigma_\pi^\mp$ decay events, and of the « stops » shown as a function of the depth in the emulsion. (Dip angle of primary restricted to $< 45°$).

should be obtained corresponding to a uniform density of $V\Sigma_\pi^\mp$ decays throughout the emulsion. As shown in Fig. 5 when this is done a definite improvement results in the shape of the distribution for the corrected $V\Sigma_\pi^\pm$ with dip $< 45°$.

7. – Scattering of Σ-hyperons.

Single scatterings of identified Σ-hyperons were recorded during the course of the experiment if 1) the angle in the horizontally projected plane was not less than 3° and 2) the space angle of the scattering was 5° or more. Hyperon primaries with a dip angle exceeding 50° and uncertain Σ^- were not included.

The track length versus energy distributions for the identified Σ^- and Σ^+ hyperons is shown in Fig. 4. The energy distribution is seen to be almost the same for both the Σ^- and Σ^+ charge identified samples. Thus the scattering data may be used for a direct estimation of the average ratio (Σ^-/Σ^+) of the scattering cross-sections.

The results for 18 cm of Σ^- and 25 cm of Σ^+-hyperon path in the energy range $(5 \div 100)$ MeV are compared with the calculated Coulomb-Rutherford scattering for a point nucleus in Fig. 6a and 6b.

The deficit of small angle scatters in the group $5° \div 10°$ is indicative probably simply of the observational loss of the smaller scatterings. Although the amount of data is small there appears to be evidence of the contribution of nuclear effects for the scattering of Σ^- with a deflection of 20° or more.

There are 5 events recorded $> 20°$, whereas the expectation from simple Coulomb-Rutherford scattering is about 1.2.

It is also of interest that the differential ratio of the scatterings (Σ^-/Σ^+) with deflection of over $10°$ is 11 Σ^- to one Σ^+ observed.

Since it appears from the present (rather insufficient) data that the Σ^- are more strongly single scattered than the Σ^+, it may then suggest that the general behaviour of the Σ^- in scattering is somewhat similar to the K^- and p^--particles. More data and calculations would however be required to test a conclusion of this kind since the difference between the distributions may be due to the effects of the charge alone.
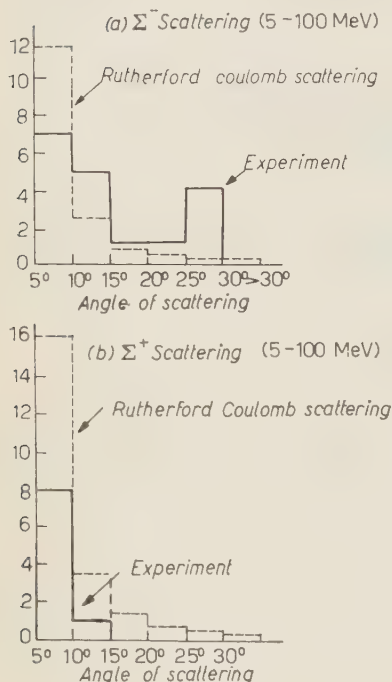
Fig. 6. – Single scatterings observed for (a) Σ^- hyperons and (b) Σ^+ hyperons. For comparison the calculated Coulomb-Rutherford scattering is shown.

8. – Non conservation of parity in Σ-hyperon decay.

From the review given in the recent Kiev Conference [24], it appears that the best measure of decay parity so far are from experiments using fast π-meson beams in hydrogen. A large asymmetry (parity violation) is indicated for the Σ_p^- decay process from the counter experiments of COOL, CORK, CRONIN and WENZEL [25]. On the other hand no asymmetry was detected for the Σ_-^- decay. The latter result is in agreement with the bubble chamber experiments [26].

Following the suggestion by GATTO (1957) [27] the method employed in nuclear emulsions assumes that a partial polarization exists for the Σ's produced from (Σ, π) events subsequent to K^--capture.

[24] D. A. GLASER: Report on Strange Particle Decays at the Kiev High Energy Nuclear Physics Conference (1959).

[25] R. L. COOL, B. CORK, J. W. CRONIN and W. A. WENZEL: Radiation Laboratory, Berkeley. Communication to D. A. GLASER (loc. cit.).

[26] E. BOLDT, H. BRIDGE, D. CALDWELL and Y. PAL: Phys. Rev. Lett.,1, 256 (1958).

[27] R. GATTO: Communication to F. M. SMITH.

Any asymmetry in the direction of the decay π-meson due to parity violation should occur about the plane determined by the (Σ, π) production. The data from the present experiment is given in Table IX. The angle θ is the

TABLE IX. – *The possible non-conservation of parity in hyperon-decay, present emulsion data.*

	θ $(0° \div 90°)$	θ $(90° \div 180°)$
$V\Sigma_p^+$	7	9
$S\Sigma_p^+$	16	13
Total Σ_p^+	23	22
$S\Sigma_\pi^+$	19	24
$V\Sigma_\pi^+$	2	2
Total Σ_π^+	21	26
$V\Sigma_\pi^\pm$	8	13

azimuth of the direction of the decay π-meson with respect to the normal ($\boldsymbol{n}$) to the production plane containing the momenta P_Σ of the hyperon and $P_{\pi assoc.}$ of the associated π-meson, as proposed by F. M. SMITH and F. E. INMAN ([28]). The direction of $\boldsymbol{n}$ is here defined as to correspond in sense to the vector $P_\Sigma \times P_{\pi assoc.}$.

Owing to the possibilities for Σ and π having collisions with nucleons the (Σ, π) plane as observed will tend to differ from the original plane of production. It is not possible to recognize all cases where it is likely to have been seriously altered. Certain instances showing evidence of marked energy exchange were however readily excluded. The average nuclear excitation energy transfer in $(\Sigma, \pi, 0)$ events was found to be about 15 MeV. (See Part I, Section 5·3, p. 714). Hence events showing exceptional energy loss (40 MeV or greater and constituting about 15% of the total) were regarded as unsuitable and have been excluded from the data of Table IX.

The contribution of the asymmetry term $\alpha \cos \theta$ should be opposite in sign for the numbers in the groups $\theta = 0° \div 90°$ and $90° \div 180°$ in Table IX. Obviously however there are too few events for the numbers to have any significance. A compilation of the world data for nuclear emulsions has been made by ALLES *et al.* (1958) ([29]) but here again it is evident that much further

([28]) F. M. SMITH and F. E. INMAN: Privately circulated letter, July 1957.
([29]) W. ALLES, N. BISWAS, M. CECCARELLI, R. GESSAROLI, G. QUARENI, Il. GÖING, K. GOTTSTEIN, W. PUSCHEL, J. TIETGE, G. T. ZORN, J. CRUSSARD, J. HENNESSY, G. DASCOLA and S. MORA: *Nuovo Cimento*, **10**, 175 (1958).

data will be necessary in order to check even the matter of greatest interest, the strongly suggested parity violation in the Σ^+ decay process indicated from the counter experiments.

<center>* * *</center>

RIASSUNTO

Questo lavoro contiene la parte III dello studio sopra l'interazione dei mesoni K⁻ in emulsione fotografica, svolto dall'European K⁻-Collaboration. Il cammino libero medio per interazione di mesoni K⁻ in volo ((10÷80) MeV) con idrogeno e con i nuclei complessi dell'emulsione viene confrontato con altri recenti dati sperimentali. Da alcuni esempi favorevoli di decadimento $\Sigma^+ \rightarrow p$ la massa dell'iperone Σ^+ viene valutata in $(2\,327.2 \pm 1)\,m_e$; da alcuni esempi di cattura a riposo di mesoni K⁻ su idrogeno viene dedotta la differenza di massa tra iperone negativo e iperone positivo $(M_{\Sigma^-} - M_{\Sigma^+}) = (14.6 \pm 1.1)\,m_e$. È stata stimata col metodo di Bartlett la vita media degli iperoni carichi: a) la miglior stima per la vita media del Σ^+ viene ottenuta usando i soli eventi in cui il protone di decadimento è emesso in avanti nel sistema del centro di massa. Si è ottenuto da 41 eventi $\tau^+ = (0.82^{+0.34}_{-0.20}) \cdot 10^{-10}$ s. b) Un valore indicativo per la vita media di un gruppo di presunti Σ^- è stato dedotto da una scelta di 70 decadimenti in volo del tipo $\Sigma^{\pm}_{\pi}$, ottenendo $\tau^- = (0.71^{+0.19}_{-0.12}) \cdot 10^{-10}$ s. Nei limiti della statistica e della selezione operata, questo valore di τ^- sembra rimanere vicino a quello di τ^+, anche in gruppi di eventi che contengono proporzioni di Σ^- molto diverse. Per quanto il valore ottenuto sia maggiore di quelli riportati in precedenti lavori in lastre nucleari sembra ancora che possa esistere un'effettiva anomalia. Il numero di interazioni secondarie prodotte da particelle cariche pesanti emesse nella cattura di mesoni K⁻ si giustifica come risultato di interazioni di protoni, con un eventuale contributo da parte di deutoni. Un sicuro esempio di interazione di Σ in volo (energia visibile 129 MeV) è stato individuato su una lunghezza di traccia di iperone di 70 cm. Il numero degli « scattering » singoli su 18 cm di traccia di Σ^- e 25 cm di Σ^+ (energie comprese tra 5 e 100 MeV) è stato confrontato con le previsioni relative allo « scattering » coulombiano da nucleo puntiforme: gli scarsi dati a disposizione, potrebbero suggerire una debole indicazione a favore dell'interazione nucleare dell'iperone Σ^-. Infine, sono stati analizzati in vista di una possibile polarizzazione del decadimento rispetto al piano di produzione della coppia (Σ, π), 45 Σ^+_p e 47 $\Sigma^{\pm}_{\pi}$ osservati in interazioni di K⁻ del tipo $(\Sigma, \pi, 0)$.

Scattering anelastico degli antiprotoni tra 30 e 250 MeV.

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Riassunto. — Lo scattering anelastico di antiprotoni, con energia compresa tra 30 e 250 MeV, contro nuclei viene studiato con la tecnica delle emulsioni. La sezione d'urto per il processo, che era già stata da noi comunicata in un precedente lavoro, viene qui rideterminata usando un diverso criterio. Si conferma però il risultato $\sigma_i = (85 \pm 20)$ mb contro il nucleo di emulsione, valore che sensibilmente differisce da quello dato da altri autori $\sigma_i = (42 \pm 11)$ mb. Ciò è dovuto ad un più efficace criterio di identificazione degli eventi basato sulla tecnica della misura delle lacune che viene qui discusso. Lo scattering anelastico viene poi analizzato con il modello ottico e si calcola un valore della sezione d'urto effettiva del $\bar{p}$ contro nucleone legato: $\sigma_{eff} = (180 \pm 50)$ mb.

1. – Introduzione.

Nelle ricerche sullo scattering anelastico degli antiprotoni contro nuclei complessi che finora sono state eseguite con la tecnica delle emulsioni nucleari [1-3], si sono individuati gli scattering anelastici per mezzo di una eccitazione nucleare visibile (uno o più rami nel punto di scattering, avendo cura di escludere gli eventi elastici $\bar{p}$-p) o per mezzo di una variazione di ionizza-

(1) W. H. BARKAS, R. W. BIRGE, W. W. CHUPP, A. G. EKSPONG, G. GOLDHABER, H. H. HECKMAN, D. H. PERKINS, J. SANDWEISS, E. SEGRÈ, F. M. SMITH, D. H. STOCK, L. VAN ROSSUM, E. AMALDI, G. BARONI, C. CASTAGNOLI, C. FRANZINETTI e A. MANFREDINI: *Phys. Rev.*, **105**, 1037 (1957).
(2) A. G. EKSPONG e B. E. RONNE: *Nuovo Cimento*, **13**, 27 (1959).
(3) G. GOLDHABER, T. KALOGEROPOULOS, R. SILBERGER: *Phys. Rev.*, **110**, 1474 (1958).

zione apprezzabile a vista. Noi pensiamo che questi metodi di individuazione non siano esenti da critiche. Infatti gli eventi così trovati sono stati 14 di cui solo 3 senza rami: mentre in questi ultimi si sono trovate variazioni di energia dal 25 al 50%, negli scattering con rami si sono trovate perdite di energia fino al 5%. Si nota cioè una notevole differenza tra la variazione di energia messa in evidenza per gli scatterings con rami e per quelli senza rami.

Abbiamo quindi deciso (nel corso di una più ampia ricerca sulle proprietà degli antiprotoni ([1])) di fare uno studio sistematico degli scattering anelastici di $\bar{p}$ nell'intervallo di energia tra 30 e 250 MeV, usando il metodo di identificazione degli scattering come è descritto nella Sezione 2.

Nella Sezione 3 elaboriamo i dati sia come avevamo fatto in un precedente lavoro ([4]), sia con un nuovo criterio ottenendo però sempre lo stesso risultato σ_i (85±20) mb. Questa sezione di urto risulta sensibilmente più grande del valore dato da precedenti autori ([2,3]) $\sigma_i = (42\pm11)$ mb e ciò è dovuto alla maggior sensibilità del metodo di identificazione degli scattering anelastici. Nella Sezione 4 infine lo scattering anelastico del $\bar{p}$ viene studiato alla luce del modello ottico e si dà così un valore della sezione d'urto effettiva del $\bar{p}$ contro nucleone legato, da cui si può dedurre una valutazione del cammino libero medio dell'antiprotone in materia nucleare.

2. – Metodo sperimentale.

Gli eventi segnalati dagli osservatori durante l'inseguimento delle tracce di $\bar{p}$ e quindi analizzati, sono tutti gli scattering. Si possono distinguere 3 tipi di eventi:

1) scattering accompagnati da più rami;

2) scattering accompagnati da un ramo con range > 20 μm e quindi sicuramente classificabili come anelastici o no. Infatti si possono escludere gli eventi che soddisfano la cinematica dell'urto elastico contro H, mentre nell'urto elastico con qualunque altro elemento, alla nostra energia, non si avrebbe mai un rinculo di tale range;

3) scattering con un ramo < 20 μm o senza ramo.

Mentre gli eventi delle classi (1) e (2) sono identificabili con scattering anelastici, la interpretazione degli eventi della classe (3) richiede uno studio dettagliato caso per caso.

([1]) E. AMALDI, G. BARONI, G. BELLETTINI, C. CASTAGNOLI, M. FERROLUZZI e A. MANFREDINI: *Nuovo Cimento*, **14**, 977 (1959).

Degli scattering non accompagnati da alcun ramo e con angolo in proiezione $\alpha > 5°$ (scelto per ragioni di efficienza di rivelazione) è stato misurato l'angolo nello spazio θ.

Si è calcolato il massimo valore di θ in funzione del range residuo di un $\bar{p}$ scatterato elasticamente da nucleo di Ag per il quale non è visibile il rinculo nemmeno di un blob ($< 1\,\mu m$).

Siccome tutti gli altri nuclei dell'emulsione sono più leggeri di Ag, tale valore deve essere interpretato come il limite superiore dell'angolo di deflessione per collisione elastica per il quale non è visibile il rinculo. Gli eventi con $\theta > \theta_{max}$ (alla corrispondente energia) sono quindi sicuramente anelastici.

In tutti gli altri casi per distinguere gli eventi elastici da quelli anelastici è necessario effettuare una valutazione della perdita di energia nello scattering attraverso misure di variazione di ionizzazione col metodo della lunghezza media dei gap o della lunghezza totale dei gap secondo i casi.

Abbiamo infatti dimostrato in un precedente lavoro (⁵) che per $r \quad R/M > 25\,\mu m/m_e$ (ove R è il range e M la massa della particella) il parametro di ionizzazione più sensibile alle variazioni di energia è la lunghezza media dei gap; per $r < 10$ è la lunghezza totale di gap, mentre per $10 < r < 25$ è indifferente l'uso dell'una o dell'altra variabile. È importante osservare che in nessun caso la densità dei gap n è una buona variabile, mentre invece è proprio questa che viene apprezzata visualmente. In particolare nella zona di $5 < r < 50$ la variazione di n è praticamente insensibile alle variazioni di r stesso mentre per $r > 20$ addirittura una perdita di energia può dar luogo ad un aumento della densità dei gap in luogo della attesa diminuzione.

Se si osserva che per la traccia antiprotonica da noi inseguita è $2 < r < 76$ si può facilmente vedere come sia essenziale effettuare la misura sistematica dei gap e non limitarsi alla valutazione visuale della variazione di ionizzazione.

3. – Risultati.

Si è analizzato un totale di 103 deviazioni di $\bar{p}$ sopra una lunghezza totale di traccia di 75.9 m di $\bar{p}$ aventi energia compresa tra 30 e 250 MeV.

Di questi scattering 2 appartenevano alla classe 1) e 4 alla classe 2) e sono quindi stati senz'altro considerati come scattering anelastici. Gli eventi studiati appartenenti alla classe 3) sono 97. Di essi 3 hanno un ramo $< 20\,\mu m$ ed è possibile attribuirli a scattering anelastici, 4 sono accettati in base alle considerazioni cinematiche svolte nel precedente paragrafo.

La distribuzione delle variazioni di ionizzazione misurata sui 94 eventi

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(⁵) C. CASTAGNOLI, G. CORTINI e A. MANFREDINI: *Nuovo Cimento*, **2**, 301 (1955).

rimanenti è riuscita asimmetrica e noi attribuiamo tale asimmetria alla sovrap-
posizione di scattering anelastici sopra una distribuzione gaussiana di eventi
elastici (come ci ha suggerito Ekspong).

Utilizzando la relazione ionizzazione-range calibrata su questo pacco di
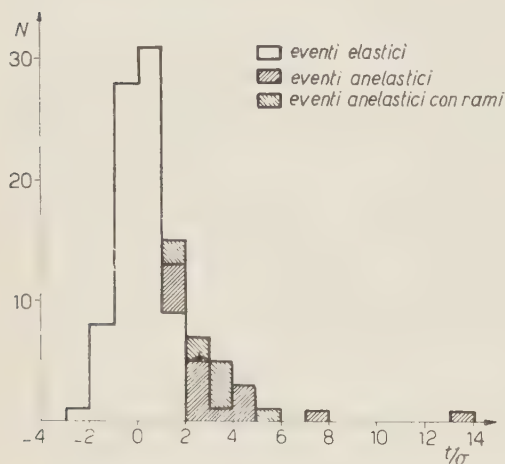emulsione si è determinata la perdita relativa di energia $t = \Delta T/T$.

In Fig. 1 è data la distribu-
zione del rapporto t/σ_t per tutti
gli eventi misurati; σ_t indica l'er-
rore standard su t.

La distribuzione rimane mani-
festamente asimmetrica, anche
quando essa è espressa nella
nuova variabile t/σ_t; per simme-
trizzarla si debbono sottrarre gli
eventi indicati in Fig. 1 come
eventi anelastici (senza rami).
Essi sono 14.

È da osservare che questo
numero di eventi coincide con
quello che si è ottenuto appli-
cando il diverso criterio di accet-
tare come scattering anelastici

Fig. 1.

gli eventi in cui $t/\sigma_t > 1.5$, cioè tutti gli eventi in cui la perdita di energia misu-
rata è superiore ad 1.5 volte l'errore.

È da osservare che in alcuni eventi di scattering anelastico la perdita di
energia relativa risulta anche del 5%.

Se si fossero usati i criteri usuali solo 2 o 3 di questi 14 eventi sarebbero
stati segnalati.

Il numero totale di eventi anelastici trovato risulta quindi uguale a 23.

Per ricavare la sezione d'urto σ_t abbiamo valutato l'efficienza di rivelazione
e la perdita geometrica dovuta al taglio $\alpha \geqslant 5°$.

L'efficienza si assume uguale a 1 per scattering del tipo 1 e 2. Per il
caso 3 l'efficienza si è ottenuta per confronto con una misura di scattering
di diffrazione fino a 1.5° da noi eseguito [6] nello stesso intervallo di energia.
In questa misura infatti l'efficienza di rivelazione di eventi con $\alpha \geqslant 5°$ era
uguale ad 1.

In definitiva si ottiene così

$$(1) \qquad \sigma_i = (85 \pm 20) \text{ mb} , \qquad (\lambda_i = (250 \pm 60) \text{ cm}) .$$

[6] G. BARONI, G. BELLETTINI, C. CASTAGNOLI, M. FERRO-LUZZI e A. MANFREDINI:
Nuovo Cimento, **15**, 1 (1960).

Questo valore è nettamente superiore a quello $\sigma_i = (42 \pm 11)$ mb trovato precedentemente e ciò è senz'altro da collegarsi al tipo di analisi seguito.

Si può inoltre osservare che esso è ancora un limite inferiore in quanto la correzione geometrica ci permette di determinare solo il numero di scattering senza rami con $\theta \geqslant 5°$, mentre su quelli con $\theta < 5°$ non si ha alcuna informazione.

Vogliamo infine osservare che uno degli scattering anelastici ha un elettrone lento nel punto di scattering che può provenire dalla diseccitazione del nucleo. Sono stati trovati altri due eventi con un elettrone associato, ma siccome il corrispondente t/σ_t risultava uguale ad 1 la presenza di tale elettrone non ci è parsa criterio sufficiente per l'accettazione tra gli scattering anelastici, benchè la probabilità che un raggio δ od un elettrone di fondo casualmente coincida con il punto di scattering sia molto piccola (nelle nostre lastre). Così facendo ad ogni modo abbiamo ulteriormente sottovalutato la sezione d'urto.

4. – Analisi e discussione.

4·1. – Lo scattering anelastico dei $\bar{p}$ è stato analizzato da EKSPONG [2] in termini di modello ottico; la σ_i così calcolata ($\sigma_i = 45$ mb) risultava in accordo con quella sperimentale ricavata dai precedenti autori ($\sigma_i = (42+11)$ mb).

Poichè il nostro risultato sperimentale, come abbiamo già sottolineato, differisce fortemente da questo, abbiamo voluto riesaminare questa questione.

Si può esprimere σ_{tot} per un dato nucleo dell'emulsione in funzione della σ_{eff}, cioè della sezione d'urto totale antinucleone-nucleone legato supponendo uguale la σ_{eff} per $\bar{p}p$ e per $\bar{p}n$:

$$(2a) \qquad \sigma_{\text{tot}} = 2\pi \int_0^{b_{\max}} b A(b) \, db \,,$$

con:

$$(2b) \qquad A(b) = 1 - \exp\left[- \varrho_0 S(b) \sigma_{\text{eff}}\right],$$

ove b rappresenta il parametro d'urto del $\bar{p}$, $S(b) = 2 \int_0^{s(b)_{\max}} u[s(b)] \, ds$ è il cammino effettivo del $\bar{p}$ nel nucleo in funzione di b e ϱ_0 è la densità della parte centrale del nucleo.

Sempre con il modello ottico si può calcolare σ_i attraverso la formula

$$(3a) \qquad \sigma_i = 2\pi \int_0^{b_{\max}} b B(b) \, db \,,$$

dove $B(b)$ è la probabilità che un antiprotone con parametro d'urto b dia luogo ad uno scattering anelastico senza annichilare successivamente

$$(3b) \qquad B(b) = \sigma_{el}\,\varrho_0\, f\eta \int_0^{s(b)_{\max}} u[s(b)]\exp\left[-\varrho_0\sigma_{eff}\int_0^{s(b)} u[s(b)]\,\mathrm{d}s\right]\cdot$$

$$\cdot \exp\left[-\varrho_0(\sigma_{ann} + \sigma_{c.e.})\int_{s(b)}^{s(b)_{\max}} u[s(b)]\,\mathrm{d}s\right].$$

La costante f tiene conto del principio di Pauli ed η del fatto che l'integrazione si effettua sul cammino iniziale del $\bar{p}$.

4'2. – Per il calcolo della (2) e della (3) bisogna specificare il modello del nucleo ed è opportuno eseguire alcune semplificazioni.

Si è assunto un modello del nucleo sfumato alla Fermi:

$$(4) \qquad \varrho = \varrho_0 \left\{\exp\left[\frac{r-c}{a}+1\right]\right\}^{-1} = \varrho_0\, u(r), \qquad \text{con} \qquad c = r_0 A^{\frac{1}{3}},$$

ove la costante ϱ_0 è aggiustata a dare il numero totale corretto dei nucleoni.

La scelta dei parametri a e r_0 influisce ovviamente in modo molto sensibile sui risultati del calcolo che noi abbiamo eseguito per $a = 0.65$ fermi e $r_0 = 1.25; 1.30; 1.35$ fermi.

Fig. 2.

La scelta di questo valore così elevato del parametro di sfumatura a (già usato da BJORKLUND e FERNBACH [7] nel loro studio sullo scattering degli antiprotoni) serve a tener conto del range finito dell'interazione $\bar{p}$-nucleone senza complicare il calcolo.

I valori scelti di r_0 risultano pure molto più grandi di quelli che normalmente sono dati dall'analisi dello scattering degli elettroni. Il valore 1.25 è quello già usato da Bjorklund-Fernbach. È evidente però dalla Fig. 2 (che dà σ_{tot} in funzione di σ_i) che esso non rende conto dei nostri risultati sperimentali, mentre invece un buon accordo si ottiene per $r_0 = 1.30$ fermi.
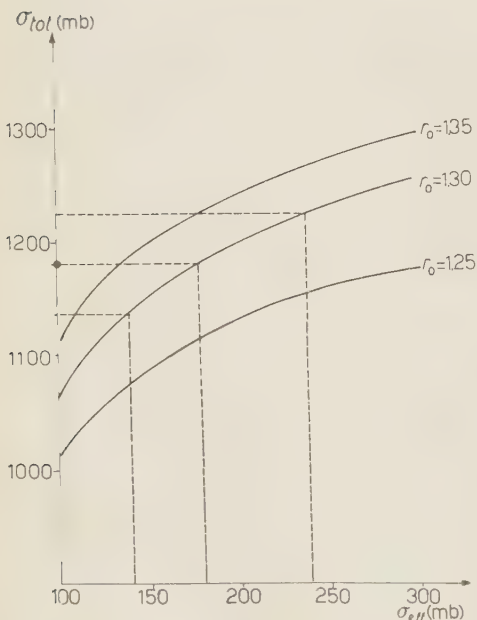
(7) F. BJORKLUND e S. FERNBACH: comunicazione privata.

Le curve di Fig. 2 sono state calcolate per $f = 1$; è da osservare che valori minori di f avrebbero comportato valori ancor più grandi di r_0.

D'altra parte questa scelta resta giustificata da considerazioni che faremo ulteriormente.

Nel calcolo della $S(b)$ si è approssimata la (4) con uno sviluppo in serie arrestato al 3° termine (*). Per semplificare il calcolo della (3) date le incertezze del modello e dei dati sperimentali a nostra disposizione si è eseguito anche l'integrale che compare nel secondo esponenziale ponendo in luogo di $\sigma_{a\,n} + \sigma_{c.e.}$ il σ_{eff} e ponendo $\eta = 1$.

Queste due approssimazioni agiscono in senso opposto e tendono a compensarsi. Con queste approssimazioni si ha:

$$B(b) = \varrho_0 f \sigma_{()} \, S(b) \, \exp \left[- \varrho_0 S(b) \sigma_{eff} \right] \,.$$

4'3. – In Fig. 3 è data la σ_{tot} contro nucleo di emulsione in funzione di σ_{eff} per $r_0 = (1.25 \text{ o } 1.3 \text{ o } 1.35)$ fermi.

Il valore sperimentale di σ_{tot} è la media dei valori finora dati e di quello da noi determinato (4) e precisamente:

$$\sigma_{tot} = \sigma_{ann} + \sigma_{c.e.} + \sigma_i = (1180 \pm 44) \text{ mb} \,.$$

Si ricava dalla Fig. 3 per $r_0 = 1.30$, che come si è visto meglio corrisponde ai dati sperimentali,

$$\sigma_{eff} = (180 + 50) \text{ mb} \,.$$

Si può osservare che da questo valore di σ_{eff} si può ricavare una valutazione del cammino libero medio del $\bar{p}$ in materia nucleare:

$$\lambda_{(\bar{p})} = \frac{1}{\sigma_{eff}} = 0.6 \cdot 10^{24} \text{ nucleoni/cm}^2 \,,$$

che risulta in accordo con quello da noi determinato con altri metodi (4).
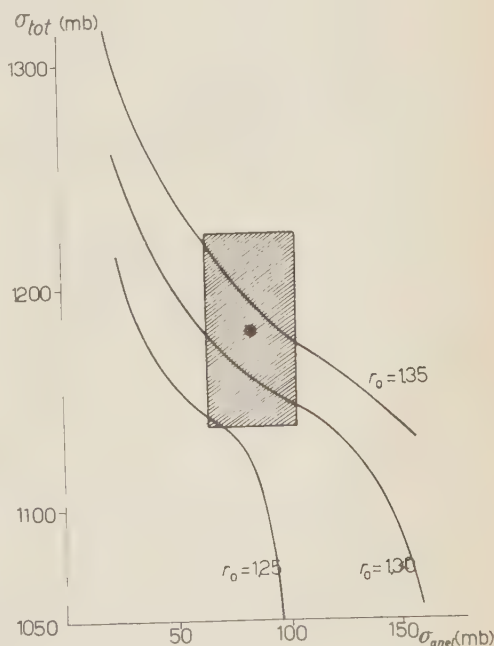

Fig. 3.

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(8) J. W. CRONIN, R. COOL e A. ABASHIAN: *Phys. Rev.*, **107**, 1121 (1957).

(9) O. CHAMBERLAIN, G. GOLDHABER, L. JUNEAU, T. KALOGEROPOULOS, E. SEGRÈ e R. SILBERGER: *Phys. Rev.*, **113**, 1615 (1959).

In Fig. 4 si è mostrata la probabilità di interazione per un $\bar{p}$ in funzione del parametro d'urto b per 2 valori di σ_{eff}. Le curve superiori si riferiscono alla interazione totale, $A(b)$, quelle inferiori allo scattering anelastico $B(b)$.
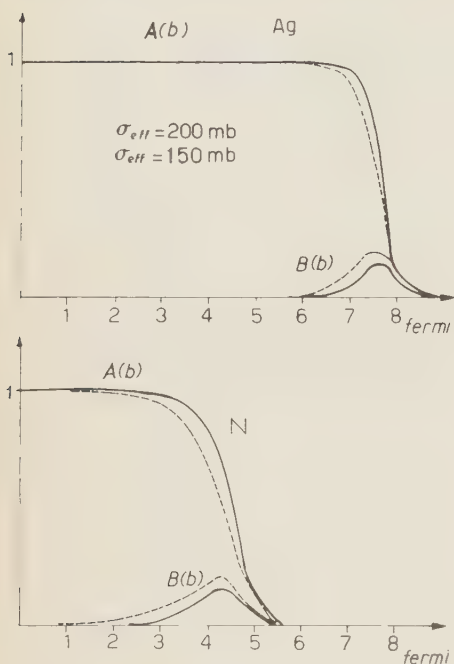
Si può vedere che: 1) aumentando σ_{eff} diminuisce la zona del nucleo interessata allo scattering anelastico e il suo contributo allo scattering stesso; 2) nei nuclei leggeri la zona del nucleo che contribuisce allo scattering anelastico è maggiore che nei nuclei pesanti.

La zona del nucleo che contribuisce a σ_i è la parte del bordo ove certamente è meno importante l'effetto del principio di Pauli. Questo può giustificare la scelta da noi fatta, $f = 1$, che influenza d'altra parte il valore calcolato di σ_i e poco quello di σ_{tot}. Anche i calcoli eseguiti da Ekspong ci sembra portino a concludere in favore di $f = 1$; sia perchè egli stesso lo verifica per la σ_{tot} sia perchè la σ_i da lui calcolata risulterebbe (con $f = 1$) ~ 75 mb in accordo con il nostro risultato sperimentale.

Fig. 4.

D'altra parte ci pare che il modello ottico, data l'incertezza dei parametri che lo caratterizzano, non permetta, soprattutto nel caso dei $\bar{p}$, una determinazione di f veramente significativa.

SUMMARY

The anelastic scattering of antiprotons against emulsion nuclei is here studied, in the energy range from 30 to 250 MeV. The cross-section for this process, already given by us in a previous paper, is redetermined on the basis of a different criterium. The result $\sigma_i = (85 \pm 20)$ mb is nevertheless confirmed, a value which appreciably differs from that of other authors: $\sigma_i = (42 \pm 11)$ mb. This is due to a more efficient criterium for the identification of the events, based on gap measurements, described in this paper. The anelastic scattering is then analyzed by means of the optical model, and a value can be given for the effective cross section of $\bar{p}$ against bound nucleons: $\sigma_{\text{eff}} = (180 \pm 50)$ mb.

On Possible Experimental Tests
for the Paradox of Einstein, Podolsky and Rosen.

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Summary. — In order to avoid the well known paradox of Einstein, Podolsky and Rosen, some modifications can be introduced in quantum mechanics, along lines pioneered by Furry. These modifications obviously have to be tested experimentally. Some time ago, Bohm and Aharonov claimed that an experiment of Wu and Shaknov (on the correlation of polarizations of annihilation photons) can be considered as an empirical proof against the Furry hypothesis. However, a careful analysis of the properties of photons shows that they are not suitable to formulate the paradox of Einstein, Podolsky and Rosen, so that the argument of Bohm and Aharonov against the Furry hypothesis is not valid. Some other possible experimental tests are proposed.

1. – Introduction.

Since the formulation of Quantum Mechanics, its assertion that only statistical predictions can be made concerning the results of measurements has been repeatedly attacked. The principal objective of these attacks was to show that the state of a system can be described in greater detail than is done by the quantum-mechanical state function [1].

One of the most widely known of these objections is the so-called paradox of EINSTEIN, PODOLSKY and ROSEN [2]. Its original formulation involved the

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[1] I. I. ZINNES: *Am. Journ. Phys.*, **26**, 1 (1958).
[2] A. EINSTEIN, B. PODOLSKY and N. ROSEN: *Phys. Rev.*, **47**, 777 (1935; hereafter) referred to as EPR.

use of continuous variables (cartesian co-ordinates and momenta) and was rapidly refuted by BOHR ([3]). Recently, BOHM and AHARONOV ([4]) showed that continuous variables are not appropriate to test the EPR paradox, and that the best way to make such a test is to use discrete quantities, such as spins.

Bohm's formulation of the EPR paradox runs as follows ([5]): one considers a system of total spin zero consisting of two particles, each of spin one-half. The spin state of the system is therefore described by the function

$$(1) \qquad \psi = 2^{-\frac{1}{2}} [\psi_+(A)\,\psi_-(B) - \psi_-(A)\,\psi_+(B)],$$

where $\psi_+(A)$ refers to the state in which particle A has spin $+\hbar/2$, etc. Note that ψ is *invariant under spatial rotations*. The two particles are then separated by a method that does not influence their total spin. After they have been separated enough so that they cease to interact, any desired component of the spin of A, $S_z(A)$ say, is measured. Then, because the total spin is still zero, it can immediately be concluded that $S_z(B) = -S_z(A)$.

Since, by hypothesis, the two particles no longer interact (unless we assume *instantaneous* hidden interaction between A and B), we have obtained a way of measuring an arbitrary component of the spin of B without in any way disturbing this particle. However if, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, this value should be considered as an *element of reality* ([2]). For instance, $S_z(B)$ is now an element of reality.

If this is true, however, it can reasonably be stated that this element of reality must have existed in B even *before* the measurement of $S_z(A)$ took place. For since there is no interaction with particle B, the process of measurement cannot have affected this particle in any way. (This is the point of view of EPR, and the root of their paradox ([2]). A different point of view was held by BOHR ([3]), according to whom particles A and B, together with the measuring apparatus form one single indivisible combined system, so that it is the measurement performed on A which creates the corresponding element of reality of B. We shall return to this question in Section **3**.)

Anyhow, before the measurement of $S_z(A)$ has taken place, we are free to choose *any* direction as the one in which the spin of particle A (and therefore of particle B) will become definite. Since this can be accomplished without in any way disturbing B, we conclude (if EPR's interpretation is admitted)

([3]) N. BOHR: *Phys. Rev.*, **48**, 696 (1935).
([4]) D. BOHM and Y. AHARONOV: *Phys. Rev.*, **108**, 1070 (1957).
([5]) D. BOHM: *Quantum Theory* (New York, 1951). Chap. XXII.

that precisely defined elements of reality must exist in particle B, corresponding to the simultaneous definition of all three components of its spin.

Now, as the wave function can specify, at most, only one of these components at a time with complete precision, we are thus led to the conclusion that the wave function does not provide a complete description of all the elements of reality existing in B. Obviously, there is no such difficulty in Bohr's interpretation.

A possible resolution of this paradox, different from that of BOHR, was proposed by FURRY [6] soon after its original formulation by EPR. To illustrate Furry's hypothesis in terms of our problem [4], we may consider the possibility that after the system of spin zero decomposes, its spin state is eventually no longer described by eq. (1) which, because of its invariance under spatial rotations, implies the puzzling correlations of the spins of the two particles. Instead, we suppose that in any individual case, the spin of each particle becomes definite in *some* (unpredictable) direction, while that of the other particle is opposite. The wave function will be the product

$$(2) \qquad \psi = \psi_{+,\theta\varphi}(A)\,\psi_{-,\theta\varphi}(B)\,,$$

where $\psi_{+,\theta\varphi}(A)$ refers to the state in which particle A has spin $+h/2$ in the direction given by θ and φ. In other words, each particle goes into a definite spin state, while the fluctuations of the other two components of the spin of one particle are uncorrelated to the fluctuations of these components of the spin of the other particle. In order to retain spherical symmetry and angular momentum conservation in the statistical sense, we can suppose that in a large aggregate of similar cases, there is a uniform probability for any direction of θ and φ.

If we admit this interpretation of the Furry hypothesis, the EPR paradox no longer exists, because there is no assurance that one will always find $S_z(A) + S_z(B) = 0$. Clearly, this last point has to be investigated *experimentally*, in order to decide whether modifications, such as Furry's hypothesis, have to be brought to current quantum theory.

BOHM and AHARONOV claim in their paper [4] that the experiment performed some time ago by WU and SHAKNOV [7] (on the correlation of the polarizations of annihilation photons) can be considered as empirical evidence against the Furry hypothesis, and thus as an empirical proof that the paradoxical aspects of the quantum theory discussed by EPR represent real properties of matter.

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[6] W. H. FURRY: *Phys. Rev.*, **49**, 393, 476 (1936).

[7] C. S. WU and I. SHAKNOV: *Phys. Rev.*, **77**, 136 (1950).

However, photons are quite different in many respects from other particles, and one must be very careful in generalizing the previous argument to their case. We here intend to show that it is indeed possible to formulate an argument similar to that of BOHM [5], involving the linear *and* the circular polarization of the photons. (*Both* kinds of polarization must be *simultaneously* considered, and not only linear polarization as was done by BOHM and AHARONOV [4]). However, it will be shown that this new formulation does not lead to a paradox, but to an inconsistency. It will then be inferred that there cannot exist a satisfactory formulation of the EPR paradox for photons.

Thus, Furry's hypothesis still awaits an experimental test. Some adequate experiments will be suggested in the last section of this paper.

2. – Analysis of the Wu-Shaknov experiment.

The annihilation radiation of positronium in its ground state (which is a $^1S_0$ state) consists of two photons that are always emitted with equal and opposite momenta, and in such a way that each photon is in a state of polarization orthogonal to that of the other, no matter what may be the choice of axes with respect to which the state of polarization is expressed. The polarization state of the resulting photons is thus described by the function [8]

$$(3) \qquad\qquad \psi = 2^{-\frac{1}{2}}(\psi_{++} - \psi_{--}) \,,$$

where ψ_{rr} is the state in which both photons have right circular polarization, etc. Taking the direction of the momenta of the photons as the z axis, this can also be written as [4]

$$(4) \qquad\qquad \psi = 2^{-\frac{1}{2}}(\psi_{xy} - \psi_{yx}) \,,$$

where ψ_{xy} is the state in which the first photon is polarized along the x axis and the second one along the y axis. This equation, like eq. (1), is invariant under rotations in the xy plane, and one sees that we have essentially here the same kind of correlation in the properties of distant particles as discussed in the introduction.

Corresponding to Furry's hypothesis, one has now two essentially different possibilities, referred to by BOHM and AHARONOV [4] as cases B_1 and B_2.

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[8] J. M. JAUCH and F. ROHRLICH: *The Theory of Photons and Electrons* (Cambridge, Mass., 1955), p. 282. The notation of JAUCH and ROHRLICH for the circular polarization of a photon directed in the negative z direction is opposite to the notation of ref. [4].

Case B_1. Each photon becomes circularly polarized about its direction of motion, but the two photons are oppositely polarized. The wave function is either

$$(5) \qquad\qquad\qquad w_{+r} \quad \text{or} \quad \psi_{--}$$

with equal probabilities for both cases.

Case B_2. Each photon goes into a state of linear polarization in some direction, while the other goes into the state of perpendicular polarization. Over many cases, one obtains the same probability for an arbitrary direction of polarization of any one of the photons. The wave function is therefore

$$(6) \qquad \psi = \psi_{xy} \cos \alpha - \psi_{yx} \sin \alpha \,,$$
$$= \tfrac{1}{2}[(\psi_{++} - \psi_{--})(\cos \alpha + \sin \alpha) + (\psi_{-+} - \psi_{+-})(\cos \alpha - \sin \alpha)] \,,$$

where α is unpredictable.

Let us note that Furry's hypothesis in this case would mean a much more serious breakdown of quantum electrodynamics than implied in the paper of BOHM and AHARONOV. Indeed, the wave function (3) can be shown to be the only possible one having the correct parity, *i.e.* the same parity as the positronium ground state [8]. Wave functions such as (5) or (6) are mixtures of terms of different parities [8], and therefore would imply *parity non-conservation in electromagnetic interactions*. In fact, the functions ψ_{-+} and ψ_{--} would also imply non-conservation of the *angular momentum* in individual processes, because they correspond to $J = \pm 2$ (see ref. [8]). Nevertheless, as already stated by BOHM and AHARONOV [4], the angular momentum would be conserved in the average over many processes. However, there cannot be some kind of « statistical conservation » of the parity, because the initial state has negative parity, while any state function for the two photons, other than the correct one, contains a part of positive parity.

Now, quantum electrodynamics allows experiments of the highest accuracy and surely is one of the most carefully tested physical theories. If parity were not conserved, this would certainly have been remarked since long. It follows that the Furry hypothesis could be rejected, in this case, from the beginning, without any reference to the Wu-Shaknov experiment.

BOHM and AHARONOV have also considered the intermediate case of elliptic polarization, and compared the suggested theories (along with standard quantum electrodynamics, which they call case A) with the result of the experiment of WU and SHAKNOV [7]. This experiment was aimed at testing whether there really is a correlation in polarization direction of the type described in

the foregoing, and it consisted in measuring the relative rate in the scattering of two photons in perpendicular planes. BOHM and AHARONOV performed the needed computations according to the Klein-Nishina formula [9], and found that the experimental result agreed with case A (standard quantum electrodynamics) and was incompatible with cases B_1 and B_2 (Furry's hypothesis). They concluded that the Wu-Shaknov experiment can therefore be considered as a proof that the paradoxical aspects of the quantum theory discussed by EPR represent real properties of matter.

3. – Photons and the EPR paradox.

However, BOHM and AHARONOV [4] have overlooked the important fact that the polarization of photons is physically quite different from the spin of fermions: as photons have zero rest mass, their spin (circular polarization) is always oriented in the direction of their propagation [10]. The components of the spin orthogonal to the direction of propagation are not gauge-invariant, [11] and therefore have no physical meaning. One thus sees that the spin of photons, if taken *alone*, cannot be used for Bohm's formulation of the EPR paradox, because it has only one component.

Linear polarization *alone* (which was the one investigated by BOHM and AHARONOV [1]) is also useless, because the two *directions* of linear polarization are similar to the two *values* of the spin in some given direction. Thus, obviously, there can exist no uncertainty relations for linear polarization alone. (In fact, the two possible states of linear polarization are combinations of the two possible states of circular polarization).

The previous argument still does not mean that photons cannot be used at all in order to test the EPR paradox: there are operators (the Stokes operators Σ_k) which can be used for this purpose. They are defined by

$$\Sigma_k = \widetilde{a_1^* a_2^*}\, \sigma_k \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

where a_n^* and a_n are the creation and destruction operators of transversal photons polarized along the n-axis, and where the σ_k are the Pauli spin matrices.

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[9] W. HEITLER: *The Quantum Theory of Radiation* (Oxford, 1954), p. 217.
[10] M. FIERZ: *Helv. Phys. Acta*, 13, 95 (1940).
[11] See J. M. JAUCH and F. ROHRLICH: ref. [8], p. 40.

The Stokes operators satisfy commutation rules ([12])

$$(7) \qquad\qquad [\Sigma_i, \Sigma_j] = 2i\hbar\, \Sigma_k,$$

where $(i\,j\,k)$ is a cyclic permutation of (123). Apart from the non-essential factor 2, these commutation rules are identical with those of the three components of the spin. It can easily be shown that Σ_2 corresponds to the circular polarization of the photon ([12]), and that Σ_3 corresponds to its linear polarization. (No simple physical meaning of Σ_1 is presently known to the authors.) Thus the equation

$$[\Sigma_2, \Sigma_3] = 2i\hbar\, \Sigma_1$$

means that the circular polarization of a photon and its linear polarization cannot be simultaneously defined.

In fact, this is a rather trivial statement: if a photon is in some eigenstate of circular polarization, it obviously cannot be in an eigenstate of linear polarization, as the two eigenfunctions for each kind of polarization are linear combinations of the two eigenfunctions for the other kind of polarization.

Now, the analogy between (7) and the commutation rules for the spin components can be used to set a new formulation of the EPR paradox, formally similar to that exposed in Section 1, simply by exchanging the words « particles of spin one-half » with « photons », and « spin components » with « Stokes operators components ». We shall only transpose one of the last sentences of Section 1: « ... We conclude that precisely defined elements of reality must exist in photon B, corresponding to the simultaneous definition of all three components of its Stokes operators... ». However, such a statement is non-sensical, because if the circular polarization of a photon is precisely defined, its linear polarization obviously cannot be precisely defined, and vice versa (this is self-evident, even without reference to the commutation relations (7)).

Thus, in the case of the Stokes operators, the EPR point of view ([2]), according to which precisely defined elements of reality must have existed in photon B even before the measurement performed on photon A, *does not lead to a paradox, but to an inconsistency.*

This does not necessarily mean, however, that one must accept Bohr's interpretation ([3]), or that Bohm's interpretation ([5]) has to be rejected in the case of the spin components of particles of spin one half. In fact, it seems that the impossibility of constructing an EPR paradox for photons is connected with the impossibility of describing them without second quantization,

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([12]) See J. M. JAUCH and F. ROHRLICH: ref. ([8]), p. 45.

while *this paradox can be raised only for that portion of quantum mechanics having a classical counterpart.*

Thus, Furry's hypothesis still awaits an experimental test.

4. – Some possible experimental tests.

Finally, we intend to propose a few experiments that can test the *original* Bohm's formulation of the EPR paradox ([5]).

a) *Pion decay.* Charged pions have zero spin, and decay into two particles of spin one-half. Unfortunately, the decay products have oriented spins, and therefore are not appropriate for our purpose. There still remains the (yet unobserved) direct non-radiative decay of the neutral pion into an electron-positron pair ([13]), the branching ratio of which should be about 10^{-7}, with respect to the usual decay into two photons. This experiment, involving the measurements of the spins of the electron and of the positron, is extremely simple from the purely conceptual point of view. However, its actual realization seems rather difficult.

b) *Auto-ionization.* Let us consider an excited helium atom in its $2s^2$ state. There is a possibility of non-radiative transition to the $1s^1$ state, the other electron flying away. The selection rules for this process (which is analogous to the Auger effect) are ([14])

$$\Delta S = \Delta L = \Delta J = 0 .$$

The helium nucleus having no spin, it follows that the resulting helium ion must have its spin opposite to that of the free electron. In principle, this can be tested. The main difficulty in the experiment seems to be the production of a sufficiently pure initial $2s^2$ state.

Still another possibility would be to start from a helium atom in its ground state ($1s^2$), and to knock off its nucleus with the help of a neutron of a few MeV. As this process is rather rapid, one can use the « sudden approximation » ([15]) in order to find the subsequent evolution of the electronic configuration. Neglecting any neutron-electron interaction, one sees that the spin function of the electrons is still given by eq. (1), so that, in principle, they can be used in order to test the EPR paradox.

([13]) S. D. DRELL: *Nuovo Cimento*, **11**, 693 (1959).
([14]) G. HERZBERG: *Atomic Spectra and Atomic Structure* (New York, 1944), p. 173.
([15]) L. I. SCHIFF: *Quantum Mechanics* (New York, 1949), p. 211.

c) Proton-proton scattering. Because of the Pauli principle, proton-proton scattering in the singlet state corresponds to partial waves with even l, while the triplet state corresponds to partial waves with odd l. Since the latter vanish at 90° in the center of mass system, the measurement at this angle (45° in the laboratory system) involves the singlet state only ([16]). A possible experiment would consist in further scattering both protons (the initial one and the recoil proton), in a fashion somewhat analogous to that of Wu and Shaknov ([7]), in order to examine whether their spins are actually correlated.

It seems that this last experiment is practically the easiest one.

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([16]) H. A. Bethe and P. Morrison: *Elementary Nuclear Theory* (New York, 1956) p. 95.

RIASSUNTO (*)

Allo scopo di evitare il ben noto paradosso di Einstein, Podolsky e Rosen, si possono introdurre alcune modifiche alla meccanica quantistica, seguendo la via indicata da Furry. Queste modifiche ovviamente devono essere controllate sperimentalmente. Qualche tempo fa Bohm e Aharonov affermarono che un esperimento di Wu e Shaknov (sulla correlazione della polarizzazione dei fotoni di annichilazione) può essere considerato una prova sperimentale a sfavore dell'ipotesi di Furry. Invece un'analisi accurata delle proprietà dei fotoni mostra che non sono convenienti per una formulazione del paradosso di Einstein, Podolsky e Rosen, cosicchè la tesi di Bohm ed Aharonov contro l'ipotesi di Furry non è valida. Si propongono altre possibili prove sperimentali.

(*) *Traduzione a cura della Redazione.*

Weak Global Symmetry (*).

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Summary. — The Feynman, Gell-Mann model of weak interactions is modified by the introduction of neutral « currents », both of the strangeness preserving (J) and strangeness changing (S) variety. The various currents, neutral and charged, are chosen and coupled in such a manner as to guarantee the $|\Delta T| = \frac{1}{2}$ selection rule. The J currents, charged and neutral, are taken together to form an isotopic vector. The charged S currents are taken to satisfy $\Delta S/\Delta Q = +1$. These conditions automatically impose on the S currents the property that they transform like the components of an isotopic spinor. The arbitrariness in the currents which remains at this stage is now removed by a definite choice, patterned after and meant to exploit Gell-Mann's model of global symmetry for strong baryon-pion interactions. In so far as the latter constitutes a useful first approximation to strong interaction physics, we can make certain fairly definite and verifiable predictions concerning leptonic decay of hyperons, and notably: the protons in $\Sigma^+ \rightarrow p + \pi^0$ decay should be polarized in an opposite sense from those produced in $\Lambda^0 \rightarrow p + \pi^-$ decay; and Ξ and Λ decays should show the same polarization properties. Presently known properties of Σ, Λ, and Ξ decays are well correlated in the present model.

1. – Introduction.

Weak reaction processes take place in a wide variety of forms. They have enough in common, however, to sustain the hope that all of them spring from a small and simple set of basic interactions. This view is embodied in the

notion of the universal Fermi interaction, which has come to mean a set of basic interactions which are quadrilinear and effectively local in character. Which fields are so coupled, with what strengths, and in what covariant forms, is still largely open to question.

A mode of description which is convenient for discussion of the Fermi interaction picture is this [1]: Regard each quadrilinear form as a product of bilinear forms, each of which in turn may be called a « current ». Thus the existence of μ-meson decayi mplies that an electron-neutrino current (eν) couples to a μ-meson-neutrino current ($\mu\nu$). We denote such lepton currents by $\mathscr{I}_{\pm}$ ($\mathscr{I}_{+} = \mathscr{I}_{-}^{\dagger}$ increases the charge by one unit). The existence of processes like β-decay, μ-meson capture, and π-meson decay implies that (eν) and ($\mu\nu$) currents couple to strongly interacting currents which can, e.g., transform neutron to proton. Denote these by $J_{\pm}$ ($J_{+} = J_{-}^{\dagger}$). Finally, processes such as $K^{+} \rightarrow e^{+} + \nu + \pi^{0}$, $K^{+} \rightarrow \mu^{+} + \nu + \pi^{0}$, etc., imply that (e$\nu$) and ($\mu\nu$) couple to strongly interacting currents which change strangeness and charge in the same direction by one unit: $\Delta S/\Delta Q = 1$. Call these currents $S_{\pm}$. It is still an open question whether these currents also contain terms with the property $\Delta S/\Delta Q = -1$. These would lead to reactions such as $K^{0} \rightarrow e^{-} + \nu + \pi^{+}$, $\Sigma^{+} \rightarrow n + e^{+} + \nu$, which are otherwise forbidden.

As for non-leptonic weak processes it is not at all clear that they bear any simple relationship to the kinds of interactions discussed above. However, a feeling for universality, as between leptonic and non-leptonic weak processes, has prompted a number of authors to relate these by postulating that the currents J and S, which, when coupled to $\mathscr{I}$ produce leptonic reactions, also couple to each other to produce the non-leptonic ones [2]. FEYNMAN and GELL-MANN [1] have gone even farther in the direction of a universal description of the weak interactions. They take the Hamiltonian to have the form

$$(1) \qquad H_{w} = (\mathscr{I}_{+} + J_{+} + S_{+})(\mathscr{I}_{-} + J_{-} + S_{-}) \,.$$

The JS terms here give rise to non-leptonic processes in which the strangeness changes by one unit. In order to avoid changes of two units which might come from $S_{+}S_{-}$ terms it is necessary to assume the rule for the S currents that

$$(2) \qquad \Delta S/\Delta Q = +1 \,.$$

This elegant model of the weak interactions seems, at this writing, to suffer from one major defect. There is no way to choose the currents $J_{\pm}$ and S

[1] R. FEYNMAN and M. GELL-MANN: Phys. Rev., 109, 193 (1958).
[2] M. GELL-MANN and A. H. ROSENFELD: Ann. Rev. Nucl. Sci., 7, 407 (1957). The earlier literature can be traced back from this source.

such that their mutual interaction leads to the $|\Delta T| = \frac{1}{2}$ selection rule [3 4]. Nevertheless, this rule appears at present to be consistent with all experiments and, beyond mere consistency, to be even somewhat compelling. Although one can regard this as being accidental—the evidence is not after all overwhelming—it would seem to be a better idea to take this empirical rule seriously.

2. – Neutral currents.

What we propose in the present note is a modification of the Feynman, Gell-Mann theory which proceeds along the following lines. In addition to the charged currents and couplings discussed above, we introduce neutral currents J_0 and S_0, and couplings

$$(3) \qquad\qquad (J_0 + S_0)(J_0 + S_0)^\dagger \,.$$

A possible objection to such neutral currents—that their coupling to corresponding neutral lepton currents, e.g., (e^+e^-), would lead to unobserved processes such as $K^0 \to e^+ + e^-$ —we deal with simply. We suppose that neutral lepton couplings don't exist. In view of the various restrictions which anyhow have to be placed on the currents in the models under discussion, this does not represent any additional violation of the notion of universality.

We shall require, for reasons of economy, that the neutral and charged currents J together form an isotopic vector. We continue to insist that the currents $S_\pm$ satisfy $\Delta S/\Delta Q = 1$. Finally we demand that the couplings responsible for weak non-leptonic processes,

$$(4) \qquad\qquad H' = J_+ S_- + J_0 S_0 + \text{h. c.} \,,$$

shall transform like the components of an isotopic spinor; i.e., we build into the model the $|\Delta T| = \frac{1}{2}$ selection rule.

It is easy enough to satisfy all of these requirements simultaneously, with considerable arbitrariness still remaining in the structure of the various currents. However, it turns out that the currents S must now themselves necessarily transform like the components of an isotopic spinor. This possibility

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[3] The evidence and theory is reviewed by R. H. DALITZ: *Rev. Mod. Phys.*, **31**, 823 (1959).

[4] See also R. L. COOL, B. CORK, J. W. CRONIN and W. A. WENZEL: *Phys. Rev.*, **114**, 912 (1959).

has been discussed before, in other contexts; and, as is well known, it implies certain definite and verifiable consequences ([5]). Here this behavior is a deduction of our model.

Finally, in order to remove the remaining arbitrariness in our currents, we shall take a major further step. We shall impose a kind of global symmetry which is the analogue for weak interactions of the strong interaction global symmetry postulated by GELL-MANN ([6]). To the extent that the latter is a good approximation to strong interaction physics, we are led to certain fairly definite predictions. In Gell-Mann's scheme, all baryons couple to pions in a highly symmetric way. These symmetries are broken by couplings, supposed weaker, to K mesons. The degree to which strong global symmetry is a useful approximation for processes not explicitly involving K mesons is, as we understand it, still an open question experimentally. For the purposes of this paper, we are supposing the approximation to be a good one.

Restricting ourselves always to bilinear forms for our currents, we observe that the currents J_i $(J_{\pm} = (1/\sqrt{2})(J_1 \pm iJ_2), J_0 = J_3)$ can be formed from a combination of the terms

$$(5) \qquad J_i: \quad \overline{N}\tau_i N, \quad \overline{\Xi}\tau_i\Xi, \quad \overline{\Sigma}_i\Lambda, \quad \overline{\Lambda}\Sigma_i, \quad -i\varepsilon_{ijk}\overline{\Sigma}_j\Sigma_k,$$

where

$$N = \begin{pmatrix} p \\ n \end{pmatrix}, \qquad \Xi = \begin{pmatrix} \Xi^0 \\ \Xi^- \end{pmatrix}.$$

We have omitted here possible contributions from π and K mesons. This is because nothing in our later arguments will require the presence of such terms. As for the currents S_- and S_0 $(S_+ = S_-^{\dagger})$, our requirements that $\Delta S/\Delta Q = 1$ and that H' transforms like an isotopic spinor lead to

$$(6) \qquad \begin{pmatrix} S_- \\ \sqrt{2}\,S_0 \end{pmatrix}: \quad \begin{pmatrix} \overline{\Lambda}p \\ -\overline{\Lambda}n \end{pmatrix}, \quad \begin{pmatrix} \overline{\Xi}^-\Lambda \\ \overline{\Xi}^0\Lambda \end{pmatrix}, \quad \begin{pmatrix} \sqrt{2}\,\overline{\Sigma}^-n + \overline{\Sigma}^0 p \\ \overline{\Sigma}^0 n - \sqrt{2}\,\overline{\Sigma}^+ p \end{pmatrix}, \quad \begin{pmatrix} \sqrt{2}\,\overline{\Xi}^0\Sigma^+ - \overline{\Xi}^-\Sigma^0 \\ \overline{\Xi}^0\Sigma^0 + \sqrt{2}\,\overline{\Xi}^-\Sigma^- \end{pmatrix},$$

where a possible (K, π) contribution is omitted, since, again, nothing in the later argument requires it.

([5]) S. OKUBO, R. E. MARSHAK, E. C. G. SUDARSHAN, W. B. TEUTSCH and S. WEINBERG: *Phys. Rev.*, **112**, 665 (1958). This rule implies a definite relation between the K_0^2 and K^+ decay rates: $R(K_0^2 \to \pi^{\pm} + e^{\mp} + \nu) = 2R(K^+ \to \pi^0 + e^+ + \nu)$, and similarly for μ-meson decay modes.

([6]) M. GELL-MANN: *Phys. Rev.*, **106**, 1296 (1957); also J. S. SCHWINGER: *Ann. Phys.*, **2**, 407 (1956).

One can now verify directly that each of the terms in S_- and S_0 transforms like an isotopic spinor, that H' has this same property, and that the charged currents S satisfy our requirement $\Delta S/\Delta Q = 1$. One could supply (π, π) and (K, K) terms in (5) and a (K, π) term in (6) such as to leave these properties unchanged. Regarding spatial transformation properties, we suppose that all the currents are vector and axial vector in character, though at this stage the relative amounts of each could vary from term to term in (5) and (6). Likewise, the relative strengths with which the various currents contribute is still open at this stage.

We shall now take our major step and assume that all of the terms in (5) have exactly the same covariant form and that they all contribute with equal strength to form the current J_i; we make a similar assumption for the terms in (6) (the relative amounts of vector and axial vector could however differ as between (5) and (6)). We then obtain the unique expressions ([7])

$$(5') \qquad J_i = \sqrt{g}\,\{\bar{N}\tau_i N + \bar{\Xi}\tau_i\Xi + \bar{Y}\tau_i Y + \bar{Z}\tau_i Z\}\,,$$

$$(6') \qquad \begin{pmatrix} S_- \\ S_0 \end{pmatrix} = \sqrt{g'}\begin{pmatrix} \sqrt{2}\,(\bar{Z}N + \bar{\Xi}Y) \\ (\bar{\Xi}Z - \bar{Y}N) \end{pmatrix}\,,$$

where

$$(7) \qquad Z = \begin{pmatrix} \dfrac{\Lambda^0 + \Sigma^0}{\sqrt{2}} \\ \Sigma^- \end{pmatrix}\,, \qquad Y = \begin{pmatrix} \Sigma^+ \\ \dfrac{\Lambda^0 - \Sigma^0}{\sqrt{2}} \end{pmatrix}\,,$$

This choice is evidently motivated by a desire to supply a maximum of symmetry in the weak Hamiltonian and is patterned after Gell-Mann's model of global symmetry for the strong interactions.

One readily verifies now that our weak Hamiltonian H' is invariant under two separate sets of operations: A) The simultaneous interchanges $Z \rightleftarrows \Xi$ and $N \rightleftarrows Y$. B) The simultaneous interchanges $Z \to N$, $N \to -Z$, $\Xi \to Y$ and $Y \to -\Xi$, followed by rotation through $180°$ around the y-axis in isotopic spin space, Y and Z being regarded as isotopic doublets.

([7]) One could imagine completing J_i by adding a vector pion current $2i\varepsilon_{ijk}\pi_k(\partial\pi_j/\partial x_\mu)$ so that, in the strong global symmetry approximation, J_i would be a conserved current. Thus, one could incorporate in an approximate manner here the Feynman, Gell-Mann explanation of the lack of renormalization of the vector coupling constant in β-decay.

Owing to the intervention of strong interactions in all weak processes (with the exception of μ-meson decay), these symmetries are useful to us only to the extent that they are shared in good enough approximation by the strong interactions. This will be the case if Gell-Mann's strong global symmetry is a valid first approximation to strong interaction effects when K mesons are not explicitly involved. With no further apologies let us assume this to be the case and ask what the consequences are [8], remembering however that what follows must be regarded as second order in weak speculations.

3. – Consequences.

3˙1. *Non-leptonic hyperon decay.* – Denote by A_s, A_p the amplitudes for hyperon decay into nucleon and pion, where A_s and A_p refer respectively to S-wave, P-wave emission. Adopting a convenient device introduced by GELL-MANN and ROSENFELD [2], we regard $A = (A_s, A_p)$ as a vector in the S-wave, P-wave plane. Let A_1, A_2, A_3, A_4, A_5 refer respectively to the processes $\Lambda^0 \to p + \pi^-$, $\Lambda^0 \to n + \pi^0$, $\Sigma^+ \to p + \pi^0$, $\Sigma^+ \to n + \pi^+$, $\Sigma^- \to n + \pi^-$. We assume time reversal invariance and neglect the small final state interaction phase shifts, so that the vectors A are taken to be essentially real. From the $|\Delta T| = \frac{1}{2}$ selection rule, which our model has been constructed to obey irrespective of the validity of strong global symmetry, we have

$$(8) \qquad\qquad A_1 = -\sqrt{2} A_2 ,$$

$$(9) \qquad\qquad \sqrt{2} A_3 = A_1 - A_5 .$$

From the additional regularities which follow from strong and weak global symmetry, one easily shows that

$$(10) \qquad\qquad \sqrt{2} A_1 = A_3 + A_4 .$$

On the experimental side, COOL *et al.* ([4]), have found that $\Sigma^+ \to p + \pi^0$ decay is characterized by a large intrinsic « up-down » asymmetry, *i.e.*, the ratio A_p/A_s is in magnitude close to unity (within a factor of about two on either side). For $\Sigma^+ \to n + \pi^+$ decay the asymmetry is at most very small. As for $\Sigma^- \to n + \pi^-$ decay, they observe no « up-down » effect, but they cannot here be sure that their Σ^- particles are polarized. If one nevertheless supposes the intrinsic effect to be small, these results, taken together with the fact that the three processes proceed at about the same rate, are consistent with the $|\Delta T| = \frac{1}{2}$ rule, Eq. (9), and imply that $\Sigma^+ \to n + \pi^+$ decay is essentially pure S-wave (P-wave), $\Sigma^- \to n + \pi^-$ decay is essentially pure P-wave (S wave), and $\Sigma^+ \to p + \pi^0$ decay has S- and P-wave amplitudes which are essentially equal in magnitude. Let us provisionally accept this characterization of Σ decays, remembering, of course, that the experimental results are consistent with, but do not prove, the $|\Delta T| = \frac{1}{2}$ rule.

We now ask what these results imply for Λ decay in our model. The implications are contained in (10) and we see that the ratio A_p/A_s for Λ decay should have the same magnitude but *opposite sign* relative to the corresponding ratio for $\Sigma^- \to p + \pi^0$ decay.

Since we know that the protons from Λ decay have left-handed polarization ([5]), the prediction is that the protons for Σ^- decay should be about equally polarized in the *right-handed sense*. Equation (10) also leads to the result that the rate for $\Lambda \to p + \pi^-$ decay should be equal to that for $\Sigma^+ \to p + \pi^0$ decay. In fact, the former rate is smaller than this by a factor of about two. However, even within the spirit of global symmetry, it is pushing matters too far to neglect here the difference in mass between Λ and Σ hyperons, hence in Q-values for Λ and Σ decay. The least arbitrary procedure is to correct the relative decay rates according to the relative sizes of phase space volume. This correction happens to supply just the factor of two needed to restore excellent agreement. At best this is partly fortuitous. In fact, one expects that, owing to centrifugal barrier effects, in Λ decay the P-wave amplitude will be reduced somewhat more than the S-wave amplitude relative to the corresponding amplitudes in $\Sigma^+ \to p + \pi^0$ decay, *i.e.*, the ratio A_p/A_s in Λ decay should be somewhat smaller in magnitude than in $\Sigma^- \to p + \pi^0$ decay. This is not inconsistent with present experimental results.

From similar arguments it follows that the Λ particles produced in decay of the cascade particle ($\Xi^- \to \Lambda^0 + \pi^-$, $\Xi^0 \to \Lambda^0 + \pi^0$) should have the same polarization as those of the nucleons produced in Λ particle decay, *i.e.*, left-

([5]) E. BOLDT, H. S. BRIDGE, D. D. CALDWELL and Y. PAL: *Phys. Rev. Lett.*, **1**, 256 (1958).

handed polarization. Without corrections for mass differences, the lifetime for $\Xi^- \to \Lambda^0 + \pi^-$ decay is predicted to be $\frac{3}{2}$ that for Λ decay. If a phase volume correction is applied, one instead finds that the two lifetimes should be essentially identical, a result which just lies within the present experimental errors—though in any event it cannot be insisted on with great precision in the present model.

3'2. *Leptonic hyperon decay.* – The relevant terms here are $(\mathscr{J}_+ S_- + \text{h. c.})$ and from (6') we observe that S_- behaves like an isotopic scalar, Y and Z being regarded as doublets. The current S is also invariant under the simultaneous interchanges $Z \rightleftarrows \Xi$, $N \rightleftarrows Y$. In the global symmetry approximation these symmetries are shared by the strong interactions. We ask for the consequences.

Let a_1 denote the full set of amplitudes—or rather, form factors [10], which characterize the process $\Sigma^- \to n + e^+ + \nu$. Similarly, let a_2 and a_3 denote the sets of corresponding amplitudes for the respective processes $\Lambda^0 \to p + e^- + \nu$ $\Xi^- \to \Lambda^0 + e^- + \nu$. One readily finds from the symmetries noted above that

$$(11) \qquad\qquad a_1 = \sqrt{2}\, a_2 = \sqrt{2}\, a_3 .$$

Similar relations, of course, hold for decay processes where electron is replaced by μ-meson. Evidently it would again be pushing matters too far to neglect mass differences among the baryons. The most reasonable way to take these into account would be to compare the corresponding form factors for the different reactions at the same invariant momentum transfer between initial and final baryon; and, in computing net decay rates, to take differences in phase volume properly into account.

Leptonic decay of hyperons offers an especially rich field for study of the structure of weak interactions, and also of the intervening strong interactions [11]. This is so because one can, in principle, determine amplitudes here over a range of electron and neutrino energies, one can detect polarization effects and correlations, etc. In this much detail the relations (11) constitute a most severe set of restrictions, despite the slight ambiguities involved in correcting them in the manner suggested to take into account baryon mass differences. If, against all reasonable hope, they were found to hold in good approximation over the full range of spectra, this would, in our opinion, constitute powerful support for the notion of combined weak and strong global symmetry.

[10] M. L. GOLDBERGER and S. B. TREIMAN: *Phys. Rev.*, **111**, 354 (1958).
[11] See C. H. ALBRIGHT: *Phys. Rev.*, **115**, 750 (1959).

Finally, we mention that on the present model the amplitudes for $\Sigma^+ \to \Lambda^0 + e^+ + \nu$ and $\Sigma^- \to \Lambda^0 + e^- + \nu$ decays should be identical and these should be smaller, by a factor of $\sqrt{2}$, than the corresponding amplitudes for neutron β decay.

RIASSUNTO (*)

Il modello di Feynman e Gell-Mann per le interazioni deboli è modificato con l'introduzione di « correnti » neutre, sia della varietà (J) (conservazione della stranezza) sia della varietà (S) (variazione della stranezza). Le varie correnti, neutre e caricate, sono scelte ed accoppiate in modo da garantire la regola di selezione $|\Delta T| = \frac{1}{2}$. Le correnti J, cariche e neutre, sono prese assieme in modo da formare un vettore isotopico. Le correnti S cariche sono prese in modo da soddisfare la $\Delta S/\Delta Q = +1$. Queste condizioni impongono automaticamente alle correnti S la proprietà di trasformarsi come componenti di uno spinore isotopico. L'arbitrarietà delle correnti che rimangono in questo stadio viene ora eliminata con una scelta definita, delineata sul modello di Gell-Mann della simmetria globale per interazioni forti barione-pione e destinata a sfruttarlo. Poichè questa simmetria costituisce un'utile prima approssimazione alla fisica delle interazioni forti, possiamo fare alcune ben definite e verificabili previsioni sul decadimento leptonico degli iperoni, e principalmente: i protoni nel decadimento $\Sigma^+ \to p + \pi^0$ devono essere polarizzati in senso opposto a quelli prodotti nel decadimento $\Lambda^0 \to p + \pi^-$; i decadimenti Ξ e Λ devono presentare le stesse proprietà di polarizzazione. Le proprietà attualmente note dei decadimenti Σ, Λ e Ξ sono ben rappresentate da questo modello.

The Analytic Properties of Perturbation Theory - II (*).

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Summary. — The definition of the physical sheet of a perturbation theory function is discussed. The types of singularity near a given surface of singularity are investigated and an expression obtained for the leading singularity. The application of these ideas to the Mandelstam representation and to further problems is indicated.

1. – Introduction.

Associated with every Feynman graph there is a function $f(z_j)$ defined by

$$(1) \qquad f(z_j) = \int_0^1 d\alpha_1 \dots d\alpha_n \frac{q(\alpha_i)\, \delta(\alpha_1 + \dots + \alpha_n - 1)}{[F'(\alpha_i; z_j)]^\varrho} ,$$

and its analytic continuation. The complex variables z_j are the scalar products of the external momenta of the graph together with the squares of the masses associated with the internal lines. The variables $\alpha_1, \dots, \alpha_n$, are the Feynman parameters associated with the graph of order n. $F'(\alpha_i; z_j)$ is the denominator that arises after symmetric integration over the internal momenta has been performed. It is a linear function of the z_j and a homogeneous function of degree one of the α_i. $q(\alpha_i)$ is the numerator and is independent of the z_j.

In a previous paper ([1]) it was shown that the singularities of $f(z_j)$ lie on the surfaces obtained by eliminating the Feynman parameters from the equations

$$(2) \qquad\qquad either \qquad \frac{\partial F'}{\partial \alpha_i} = 0 \qquad or \qquad \alpha_i = 0 \, , \qquad\qquad i = 1, \dots, n.$$

$f(z_j)$ is multivalued and not all the singularities necessarily appear on each of its sheets.

It is important, therefore, to understand what is the sheet of the function relevant for physics, and this is discussed in Section 2. In Sections 3 and 4 we seek to understand the behaviour of the function near one of the surfaces of singularities arising from a specific choice of the alternatives in equations (2). The essential behaviour is given by a singular function of the equation of the surface multiplied by an integer arising from the relevant branches of other singularities of $f(z_j)$. It is the vanishing of this integer which causes singularities to fail to appear in certain sheets. Methods for investigating the nature of the singular function associated with each surface are also discussed.

In conclusion the way in which these results could be used to investigate the kind of analytic properties conjectured by MANDELSTAM ([2]) is discussed and possible further developments of the theory outlined.

2. – The physical sheet.

In physics we are directly concerned with the function $f(z_j)$ only for real values of the z_j. The relevant branch of the function is defined by Feynman's prescription which requires the external scalar products to be real and the internal masses to be $n_i^2 - i\varepsilon$, where ε is to tend to zero through real positive values. For $\varepsilon \neq 0$ the Feynman integral is well defined for integrations along the real α-axes and so no deformation of contours as described in I is necessary until ε is actually zero. Thus only solutions of (2) which give all the α_i real and positive can produce real singularities and since the effect of $i\varepsilon$ is to change a coincident zero into two separated zeroes, one above and the other below the real axis, coincident singularities necessarily « pinch ».

These observations lead immediately to the rules obtained otherwise by J. C. TAYLOR ([3]) and LANDAU ([4]) for the location of real singularities, and in particular to the picturesque dual diagram analysis.

([1]) J. C. POLKINGHORNE and G. R. SCREATON: *Nuovo Cimento*, **15**, 289 (1960). Referred to as I.

([2]) S. MANDELSTAM: *Phys. Rev.*, **112**, 1344 (1958).

([3]) J. C. TAYLOR: *Phys. Rev.*, to be published.

([4]) L. D. LANDAU: *Nucl. Phys.*, **13**, 181 (1959).

What is desired in exploiting the analytic properties of quantum field theory is to replace the Feynman limit in terms of the internal masses by suitable limits in terms of external scalar products and eventually to treat the physical functions as limiting values of a single-valued function in the space of external scalar products suitably cut. In terms of the techniques developed in this paper this is best discussed with reference to the singular functions introduced in Section 4, whose cuts produce the necessity of defining a limiting procedure. The arguments of these functions are the algebraic equations of the singular surfaces and the possibility of exchanging a limit in terms of the internal masses for one in terms of the external scalar products will depend on how these variables appear in these equations.

3. – The general nature of the singularities.

We first discuss the simple case where there is a single variable of integration α after the δ-function has been eliminated. As $f(z_j)$ is continued along paths in the z_j space it will be necessary to deform the original contour C as described in I. This deformation is correctly prescribed by requiring that every time a singularity crosses C it is to be encircled with an additional contour C' whose sense is determined by whether the singularity crosses C from above or from below.

If a path in z_j-space encircles an end point singularity the effect of traversing it is to cause a singularity in the α-plane to encircle one of the end points of C. This singularity therefore returns to its original position encircled with an additional contour of the type C'. Encircling the end point singularity in the z_j space n times in the same sense produces n such contours C' in the same sense. Therefore the values of the function on the different sheets generated by an end point singularity differ by $n\bar{f}(z_j)$ where n is an integer and $\bar{f}(z_j)$ is the contribution from integrating round a single contour encircling the relevant singularity. End point singularities therefore generate an infinite number of sheets of the function provided the relevant singularity gives a nonzero contribution from the contour C'.

The values of the z_j specify the locations of the singularities. On encircling a coincident singularity elementary analysis shows that the two relevant singularities execute paths that cause them to change places. This may cause a change in the value of the function because on the branch in question the singularities may be surrounded by differing numbers of contours C' arising from end point singularities. This fact, that the essential behaviour of the coincident singularity depends on the relevant branches of end point singularities illustrate the behaviour of $f(z_j)$ near its singular surfaces discussed in Section 4. Clearly, on encircling twice a singularity arising from the coinci-

dence of two singularities $f(z_j)$ always returns to its original value and so those singularities generate at most two sheets of the function. Similarly singularities arising from the coincidence of more than two singularities would produce a larger number of sheets.

In this discussion we have assumed so far that the function being integrated is itself single valued. This is insufficient for our purposes since the successive integrations in (1) have integrands which are multivalued after the first integration has been performed. Each singularity has associated with it a cut and the presence of these cuts does not permit the deformations of contours to be described in terms of the circles C'. Consequently the change in $f(z_j)$ on encircling a singularity also contains a contribution from the integrals along opposite sides of the cuts and it is possible for a coincident singularity to generate an infinite number of sheets in this case.

4. – Behaviour near a surface of singularities.

Near a surface of singularities the singular part of $f(z_j)$ arises from those parts of the contours in the neighbourhood of the singular points in the x-integrations. One may expect that these parts can be cast into a standard form and the singular part of the function determined as a function of the equation of the surface. In this section we shall do this for the leading part of the singularity.

After the δ-function has been eliminated in equation (1) by integrating over x_n let us suppose that we are concerned with a singularity arising from l end points and $n - l - 1$ coincident points. We may rearrange the order of integration so that without loss of generality we may suppose that the $\alpha_{n-1}, ..., \alpha_{n-l}$ integrations have end point singularities and the $\alpha_{n-l-1}, ..., \alpha_1$ have coincident singularities. When z_j is in the neighbourhood of the surface of singularities we are concerned with the integrals in the neighbourhoods of the points $\bar{x}_i(z_j)$, where $\bar{x}$ denotes either the value of the relevant end point or the solutions of the equations

$$(3) \qquad \frac{\partial F'}{\partial \alpha_i}(\alpha_1, ..., \alpha_{n-l-1}, \bar{\alpha}_{n-l} ..., \bar{\alpha}_{n-1}; z_j) = 0 , \quad i = 1, ..., n - l - 1,$$

In order to find the leading singularity it will be sufficient to take

$$(4) \qquad F'(x_i; z_j) \sim F'_a(x_i, \ \bar{\alpha}_i; z_j)$$

$$= \bar{F}' + \sum_{j=n-l}^{n-1} \frac{\partial \bar{F}'}{\partial \alpha_j}(\alpha_j - \bar{\alpha}_j) + \frac{1}{2} \sum_{j,k=1}^{n-l-1} \frac{\partial^2 \bar{F}'}{\partial \alpha_j \, \partial \alpha_k}(\alpha_j - \bar{\alpha}_j)(\alpha_k - \bar{\alpha}_k) .$$

Barred quantities are evaluated at the points $\alpha_i = \bar{\alpha}_i$. The contours may always be straightened out to run parallel to the real axes when passing through the points $\bar{\alpha}_i$. Although we are only concerned with some finite segment of contour in the neighbourhood of $\bar{\alpha}_i$ these segments can be continued to infinity without adding to the singular part provided ϱ is sufficiently large. In consequence the form of $f(z_j)$ near the surface of singularities is given by

$$
(5) \qquad f(z_j) \sim \varepsilon \int_{-\infty}^{\infty} d\beta_1 \ \dots \ d\beta_{n-l-1} \int_{0}^{\infty} d\beta_{n-l} \ \dots \ d\beta_{n-1} \ \frac{m_1 \ \dots \ m_{n-1}}{[F_a'(\beta_i, z_j)]^{\varrho}} =
$$

$$
= \varepsilon \ \frac{m_1 \ \dots \ m_{n-1}(\sqrt{2\pi})^{n-l-1}}{\sqrt{D}\,(\partial \bar{F}/\partial \alpha_{n-l}) \ \dots \ (\partial \bar{F}/\partial \alpha_{n-1})} \cdot \frac{\Gamma(\varrho - \frac{1}{2}n - \frac{1}{2}l + \frac{1}{2})}{\Gamma(\varrho)} \cdot \frac{1}{[\bar{F}]^{\varrho - \frac{1}{2}n - l + \frac{1}{2}}} \ .
$$

where

$$
(6) \qquad\qquad D \equiv \det \left[\frac{\partial^2 \bar{F}}{\partial \alpha_i \, \partial \alpha_j} \right], \qquad\quad i, j = 1, \dots, n - l - 1;
$$

ε is $(-)^{\lambda}$, where λ is the number of upper end points; and m_i is the number of times the α_i contour passes through the point $\bar{\alpha}_i$. The result holds provided

$$
(7) \qquad\qquad\qquad \varrho - \tfrac{1}{2}n - \tfrac{1}{2}l + \tfrac{1}{2} > 0 \,,
$$

which is the condition for our manipulation of contours to be valid. However, the asymptotic form of an integral in which (7) does not hold can be obtained by observing that replacing F' by $F' + \eta$ and then differentiating a sufficient number of times with respect to η relates the first integral to one which satisfies (7). The general form of (5) continues to hold therefore, with some modification of the numerical coefficients, provided that $1/[\bar{F}]^n$ is interpreted as $(\bar{F})^{-n} \log \bar{F}$ when n is a non-positive integer. Finally we note that

$$
(8) \qquad\qquad\qquad \bar{F}(z_j) = 0
$$

is the equation of the surface of singularities.

One may expect the general form of this result to hold for the exact singularity of $f(z_j)$ in the neighbourhood of a surface of singularities. That is to say we expect that near the surface

$$
\sigma(z_j) = 0
$$

the function is given asymptotically by

$$
(10) \qquad\qquad\qquad f(z_j) \sim N \, \varphi(z_j) \, F\big(\sigma(z_j)\big) \,,
$$

where $F(\zeta)$ is a function singular at $\zeta = 0$, $\varphi(z_j)$ is a function regular in the neighbourhood of the surface, and N is an integer. The value of N depends on the branches associated with the singularities corresponding to replacing some of the coincident singularities associated with the given surface by end point singularities.

5. – The Mandelstam representation.

In order that a Mandelstam type representation should hold it is necessary that the Feynman limit defining the physical sheet should be replaceable by suitable limits in terms of the external scalar products and that the function should be regular in the space of these external scalar products suitably cut. The analysis of Sections 3 and 4 shows that the singularities of $f(z_j)$ arise from functions of the singular surfaces and knowing the equations of these surfaces enables one to determine in any particular instance whether this redefinition of the physical sheet is possible. Assuming that it is, it is then necessary to show that the singularities in the cut space are those appropriate to the desired spectral representation. The presence or absence of any singularity depends on the value of N in equation (10). One simple way of calculating this is to observe that if there is a real point on one of the sections into which the surface is divided by the cuts associated with the other relevant singularities, then the vanishing or not of N for that section depends on the absence or presence of the corresponding real singularity. This is the essential basis of the elegant analysis of the fourth order scattering amplitude given by TARSKI ([5]). It also provides a reasonably simple means of trying to find conditions under which Mandelstam's conjecture fails for higher order graphs, but a sufficient condition for the truth of the conjecture in these cases would involve an analysis so exhausting that its feasibility seems to depend on finding new properties concerning the mutual relationships of the surfaces of singularity. These might conceivably arise from considering the surfaces as the envelopes of sets of planes.

6. – Discussion.

We have tried in this paper to give a general discussion of the nature of the functions associated with perturbation theory. In terms of this picture one can readily understand the way the logarithms in the Källén-Wightman

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([5]) J. TARSKI: *Phys. Rev.*, to be published.

third order vertex function ([6]) conspire to produce an integer on the surface of anomalous singularities—a fact which at first sight seems rather mysterious.

The work of Section 3 makes it possible to determine the change in the function on encircling a given singularity. The work of Section 4 shows how the leading singularity associated with a surface can be evaluated. It is possible that a refinement and extension of these ideas might make it possible to determine fully the form of the function, for the work of Section 4 can be easily extended to give all the power-type singularities while the logarithmic singularities can be determined from the changes in the function on encircling the singularity. However, this seems too difficult to do in general at present and so some specific examples are in course of investigation.

<p align="center">* * *</p>

([6]) G. KÄLLÉN and A. S. WIGHTMAN: *Mat. Fys. Skr. Dan. Vid. Selsk.*, **1**, No. 6 (1958).

RIASSUNTO (*)

Si discute la definizione della falda fisica di una funzione in teoria della perturbazione. Si analizzano i tipi di singolarità prossimi ad una data superficie di singolarità ed una espressione ottenuta per la singolarità principale. Si indica l'applicabilità di questi concetti alla rappresentazione di Mandelstam ed a problemi ulteriori.

(*) *Traduzione a cura della Redazione.*

Haag's Theorem in a Finite Volume (*).

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Summary. — The question of extending Haag's theorem to a finite volume is explored. A positive result is suggested.

Haag's theorem [1] depends in an essential way on the uniqueness of the vacuum state in quantum field theory. This uniqueness results in the proof, from the infinite volume of space considered. If the overlap of two spatially homogeneous wave functions is not unity in a finite volume, the overlap will be zero in an infinite volume. A more interesting question is thus the overlap in a finite volume of space.

We consider a comparison of the vacua of a free field theory and a coupled field theory. The overlap of the vacua in a finite volume is not conveniently expressed in terms of vacuum expectation values of operators of the coupled field. (An arbitrarily large negative exponential of the number operator yields the square of the overlap matrix element.) It is very easy, however, to calculate the number of bare particles per unit volume in the coupled vacuum. This is expressible directly in terms of the one particle Green's function.

If one writes:

$$\langle 0 \,|\, \varphi(x)\, \varphi(0)\,|\, 0 \rangle = \frac{1}{2} \int \mathrm{d}\varrho(M) \int \frac{\mathrm{d}^3 k}{(2\pi)^3 \sqrt{k^2 + M^2}} \exp\left[i(k \cdot x - \omega t)\right]$$

[1] R. Haag: *Dan. Mat. Fys. Medd.*, **29**, No. 12 (1955).

for a scalar field, and:

$$\langle 0|\psi(x)\overline{\psi}(0)|0\rangle = -\frac{1}{2}\int d\sigma_2(M)\int \frac{d^3k}{(2\pi)^3\sqrt{k^2+M^2}}\exp\left[i(k\cdot x - \omega t)\right] +$$

$$+\frac{1}{2}\int d\sigma_1(M)\int \frac{d^3k}{(2\pi)^3\sqrt{k^2+M^2}}(M-i\gamma\cdot k)\exp\left[i(k\cdot x - \omega t)\right],$$

for a spinor field, the number of bare particles of mass M_0 per unit volume in the vacuum becomes:

$$N_B = \frac{1}{4}\int d\varrho(M)\int \frac{d^3k}{(2\pi)^3}\left|\frac{\sqrt{k^2+M^2}}{\sqrt{k^2+M_0^2}} + \frac{\sqrt{k^2+M_0^2}}{\sqrt{k^2+M^2}} \quad \text{²}\right|.$$

$$N_F = 4\int \frac{d^3k}{(2\pi)^3}\left|\int d\sigma_2(M)\cdot\frac{M_0}{2\sqrt{k^2+M^2}\sqrt{k^2+M_0^2}} + \right.$$

$$\left. + \int d\sigma_1(M)\frac{1}{2\sqrt{k^2+M^2}\sqrt{k^2+M_0^2}}\left(-M_0 M - k^2 + \sqrt{k^2+M^2}\sqrt{k^2+M_0^2}\right)\right|.$$

Thus this number operator is finite if the third moment of the Lehmann weight is finite for bosons, and never except for the free field theory of mass M_0 for fermions. Further, in the fermion case, if one changes merely the bare mass, all the states of the new theory are orthogonal to all the states of the theory of mass M_0 even in a finite volume! (In this case the overlap of states is proportional to a negative exponential of the number operator.) It seems likely that this is true for most field theories.

It would be interesting to examine self coupled boson theories in a finite volume where the states may actually be expanded in terms of the free particle states if the weight function decreases quickly enough at high masses. Of course, one would then find it difficult to maintain Lorentz invariance.

RIASSUNTO (*)

Si indaga sul problema di estendere il teorema di Haag ad un volume finito. Si suggerisce un risultato positivo.

(*) *Traduzione a cura della Redazione.*

Note on Possible Rare Decay Modes for Elementary Particles (*).

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Summary. — Some of the bound-state decay processes of elementary particles, like a neutron decaying into a hydrogen-atom, are investigated. It is found that they may happen in a rate of one to a million, compared to the usual modes.

The purpose of this note is to point out the possibility of a rare mode of decay of neutrons and other particles. A neutron (or Λ) decays into an antineutrino and a hydrogen-atom instead of proton, electron and antineutrino. Though this is very rare and may be quite difficult to detect experimentally, we report our results here in the hope that future experiments may provide a test for them.

We use the well-established $V-A$ theory [1]:

$$H_1 = \frac{1}{\sqrt{2}} G \bar{p} \gamma_\mu (1 + \gamma_5) n \cdot \bar{e} \gamma_\mu (1 + \gamma_5) \nu = - \sqrt{2} G \bar{p} (1 - \gamma_5) e_c \bar{\nu}_c (1 + \gamma_5) n,$$

where e_c and ν_c are the charge conjugate operators for electron and neutrino, respectively.

The calculation is straightforward, and we neglect the relativistic as well as mesonic effects. The transition rate for $n \rightarrow H + \bar{\nu}$ then is given by the

[1] E. C. G. Sudarsham and R. E. Marshak: *Phys. Rev.*, **109**, 1860 (1958); R. P. Feynman and M. Gell-Mann: *Phys. Rev.*, **109**, 193 (1958); J. J. Sakurai- *Nuovo Cimento*, **7**, 649 (1958).

following formula

$$w(\mathrm{n} \to \mathrm{H} + \bar{\nu}) - \frac{4}{\pi^2} \cdot \frac{G^2}{c^3 \hbar^4} \cdot \frac{E^2}{a_0^3} = 2.7 \cdot 10^{-9} \ \mathrm{s^{-1}} \ ,$$

where a_0 is the Bohr radius of the ground state of the hydrogen-atom and E the energy of the antineutrino. Thus, the branching ratio for the normal to the new modes is

$$\frac{\mathrm{n} \to \mathrm{H} + \bar{\nu}}{\mathrm{n} \to \mathrm{p} + \mathrm{e}^- + \bar{\nu}} = 2.7 \cdot 10^{-6} \ ,$$

which means that we may find our new mode in a ratio three to a million, compared to the usual decay. It may be worth-while to note that the final spin state of the hydrogen-atom is a singlet for our $V - A$ theory. The probability of creation of the hydrogen-atom for the excited states in N-wave is smaller by a factor n^{-3} (where n is the principal quantum number) compared to that of the ground state.

The same calculation is also applicable for Λ-decays, and we find

$$\frac{\Lambda \to \mathrm{H} + \bar{\nu}}{\Lambda \to \mathrm{p} + \mathrm{e}^- + \bar{\nu}} \simeq 2 \cdot 10^{-12} \ ,$$

$$\frac{\Lambda \to (\mathrm{p} + \mu^-)_B + \bar{\nu}}{\Lambda \to \mathrm{p} + \mu^- + \bar{\nu}} \sim 1 \cdot 10^{-5} \ ,$$

where $(\mathrm{p} - \mu^-)_B$ is a bound state of proton and μ-meson analogous to that of the hydrogen-atom. However, these are too small to be of any interest, since the β-decays of Λ are already quite rare.

Furthermore, we can investigate a similar calculation for K-decays (*):

$$K_2^0 \to (\pi^\pm + \mathrm{e}^\mp)_B \pm \nu \ ,$$

$$K_2^0 \to (\pi^\pm + \mu^-)_B \pm \nu \ ,$$

where $(\pi^\mp + \mathrm{e}^\mp)_B$ or $(\pi^\pm + \mu^\mp)_B$ means the bound state analogous to the hydrogen-atom. We estimate decays rates for these modes by assuming the following phenomenological interaction (²)

$$H_1 = g(\pi^* \partial_\mu K_0 - K_0 \partial_\mu \pi^*)(\bar{e}\gamma_\mu (1 + \gamma_5)\nu) + \mathrm{c.\,c.} \ ,$$

$$H_1' = g(\pi^* \partial_\mu K_0 - K_0 \partial_\mu \pi^*)(\bar{\mu}\gamma_\mu (1 + \gamma_5)\nu) + \mathrm{c.\,c.} \ .$$

(²) For example, see: M. SUGAWARA: *Phys. Rev.*, **113**, 1361 (1959).
(*) *Note added in proof.* – The process $\mathrm{K}^+ \to (\pi^+ + \pi^-)_B + \pi^+$ may be easier to be detected experimentally. The branching ratio to the normal mode is expected to be one to a million, again, in this case.

Then, we estimate that

$$\frac{K_2^0 \to (\pi^+ + e^-)_B + \bar{\nu}}{K_2^{\cdot 0} \to \pi^+ + e^- + \bar{\nu}} = \frac{K_2^{\cdot 0} \to (\pi^- + e^+)_B + \nu}{K_2^0 \to (\pi^- + e^+ + \nu)} \sim 1 \cdot 10^{-11},$$

$$\frac{K_2^0 \to (\pi^+ + \mu^-)_B + \bar{\nu}}{K_2^0 \to \pi^+ + \mu^- + \bar{\bar{\nu}}} - \frac{K_2^{\cdot 0} \to (\pi^- + \mu^+)_B + \nu}{K_2^0 \to \pi^- + \mu^+ + \nu} \sim 3 \cdot 10^{-6}.$$

* * *

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Note added in proof.

RIASSUNTO

Si studiano alcuni dei decadimenti delle particelle elementari in sistemi legati, come il decadimento del neutrone in atomo di idrogeno e neutrino. Si trova che essi possono avvenire con la probabilità di uno a un milione rispetto ai comuni modi di decadimento.

Some Integral Representations in Field Theory (*).

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Summary. — It is shown that the properties of positive mass and energy, and of causality lead in quantum field theory to problems in the theory of analytic completion. A representation for the double commutator previously proved the most general is stated, and its connection with analytic completion noted. A similar representation for the triple commutator is proved to satisfy causality and the positive mass and energy condition, but is not shown to give all the functions with these properties. A conjecture about the envelopes of holomorphy of domains occurring in the discussion of analytic properties of the four-point Green's function is made. Integral representations for the n-fold multiple commutator are proposed, and for some other combinations of Wightman functions. These representations lead to some results in analytic completion.

1. – Introduction.

In a theory of quantized fields $A(x)$, $B(x)$ ... we can replace [1] the postulate that every state has non-negative mass and energy by a requirement of regularity on the basic functions of the theory, the vacuum expectation values of products of field operators, such as $'0|A(x_1)B(x_2)...C(x_n)|0'$. This must be regular in $I'_{n-1}(y_1, ..., y_{n-1})$, where $y_i = x_i - x_{i+1}$ and I'_{n-1} is a domain of holomorphy known as the extended tube. Further, all the functions obtained from the order $AB...C$ by a permutation must be regular in the « permuted » extended tube obtained from $I'_{n-1}(y_1, ..., y_{n-1})$ by the appropriate permutation

of the x variables. Because of invariance under translations the functions depend only on the differences $y_i = x_i - x_{i+1}$.

The postulate of causality (local commutativity) is that

$$(1.1) \qquad\qquad [A(x_1), B(x_2)] = 0$$

if $x_1 - x_2$ is space-like, written $x_1 \sim x_2$. This implies that the expectation values $\langle 0 | A(x_1) B(x_2) C(x_3) | 0 \rangle$ and $\langle 0 | B(x_2) A(x_1) C(x_3) | 0 \rangle$ are equal if $y_1 \sim 0$, and, since some of these points are regular points of both functions, they continue one another as one regular function of the complex variables y_1, y_2, the functions ABC and BAC for real y being obtained as the limit to various parts of the boundary. Jost [2] proved that $I'_n(y_1, ..., y_n)$ contains the real points $y_1, ..., y_n$ if and only if

$$(1.2) \qquad\qquad \left(\sum_i \alpha_i y_i\right) \sim 0 \qquad\qquad \text{for all } \alpha_i > 0.$$

Such points are called Jost points (J.P.). A refinement of the theory which is becoming important is to note the difference between local commutativity (1.1) and Jost point commutativity (J.P.C.), which requires the various permutations of a given order to be equal *somewhere* where they are regular, say at Jost points (1.2). This is a rather less strong assumption than that they should be equal for all space-like differences, as in (1.1). It follows from the work of KÄLLÉN and WIGHTMAN that, for products of three fields, J.P.C. implies commutativity [*]. We shall conjecture a theorem below which implies this for the vacuum expectation value of a product of any number of fields [*], and therefore gives the operator condition (1.1), since, as WIGHTMAN has shown, a field theory is completely specified by its set of vacuum expectation values.

The postulate of J.P.C. implies that the $n!$ products of n fields are limits of one regular function in the union of the $n!$ permuted extended tubes, which are equal in pairs by the CPT theorem [2]. Conversely, any function regular in this union satisfies J.P.C. But it can be shown that any function regular in such a domain D is also regular in a larger domain, the envelope of holomorphy of D (E.H. of D). Before progress in the theory can be made this envelope of holomorphy must be found. The methods available for finding envelopes of holomorphy, when applied to this problem [1], lead to rather difficult manipulations in which there is no systematic way of proceeding; moreover the method used in [1] is insufficient in principle to solve the problem for products of four operators. In a previous paper [3] the author showed that,

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[*] *Note*: D. RUELLE has now proved this.

[2] R. JOST: *Helv. Phys. Acta*, **30**, 409 (1957).

[3] R. F. STREATER: *Proc. Roy. Soc.*, to be published.

by using the method of integral representations, a technique more suited to the special problem can be developed. Although the method is not sufficiently general to solve the whole problem in the three-fold case the treatment is simplified in certain parts. More important, the integral representation provided a systematic method for finding the envelope of holomorphy of certain domains.

The representation for the double commutator proved [3] previously can be generalised almost on sight to n-fold multiple commutators [4], but without the proof that the representation is the most general. It is hoped that, nevertheless, the formulae will prove to be of use as a guide to solving the problem in the general case.

2. – The double commutator.

In this section we state the representation for the double commutator proved in [3, 4], to be the most general.

Notation:

$$x = x_1 - x_2, \qquad y = x_2 - x_3, \qquad z = x_3 - x_1 ; \qquad x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 .$$

$$\overline{D}(p, q) = \int \exp\left[ipx + iqy\right] \mathrm{d}^4x \, \mathrm{d}^4y \, D(x, y) .$$

If $x^2 > 0$, $x_0 > 0$ we shall write $x > 0$. If $x^2 > 0$, $x_0 < 0$ we write $x < 0$, If $x^2 < 0$ we write $x \sim 0$. $A(x_1)$, $B(x_2)$, $C(x_3)$, ... will be shortened to $A, B, C,...$

Theorem [3]. – Suppose

$$D(x, y) = \langle 0 \,|\, \left[A(x_1), \, [B(x_2), \, C(x_3)]\right] \,|\, 0 \rangle$$

has the following properties

(2.1) $$D(x, y)$$

is a Lorentz invariant tempered ditribution.

(2.2) $$D(x, y) = D(-x, -y) ,$$

(2.3) $$D(x, y) = 0 \quad \text{if } y \sim 0 ,$$

(2.4) $$\overline{D}(p, q) = 0 \quad \text{if } p \sim 0 \text{ or if } q \text{ and } p - q \sim 0 .$$

[4] R. F. STREATER: *Ph.D. thesis* (London, 1959).

Then $D(x, y)$ has the representation

$$(2.5) \qquad D(x, y) = \int_0^\infty \varphi(s, t, \lambda; k)\, \Delta_2(x,\, x+y;\, s, t, \lambda)\, \Delta_1(y; k)\, \mathrm{d}s\, \mathrm{d}t\, \mathrm{d}\lambda\, \mathrm{d}k\,,$$

for some φ.

We remark that the double commutator has the properties (2.1)–(2.4), and therefore has the form (2.5). The double commutator, and also the representation (2.5), has the additional property

$$(2.6) \qquad\qquad\qquad\qquad D(x, y) = 0 \qquad\qquad\qquad \text{if } x \text{ and } x+y \sim 0$$

Therefore (2.6) is implied by (2.1)–(2.4).

In (2.5), we define

$$(2.7) \quad \Delta_2\,(x, y;\, s, t, \lambda) =$$

$$= \int \delta(p^2 - s^2)\, \delta(p \cdot q - \lambda)\, \delta(q^2 - t^2)\, \varepsilon(p_0)\, \exp\left[-\,(ipx + iqy)\right] \mathrm{d}^4 p\, \mathrm{d}^4 q\,.$$

That (2.5) has the property (2.6) is proved similarly to the proof of (3.5) below.

Let us for the moment assume that J.P.C. and commutativity are equivalent. It has been shown ([3]) that from the representation (2.5) we can construct the most general form for a function regular in $I_2'(x, y) \cup I_2'(x+y, -y)$ and this function turns out to be regular in $I_2'(x, x+y) \cap I(\tfrac{1}{1}y)$, but may have singularities on the boundary. Therefore, *any* function regular in $I_2'(x, y) \cup I_2'(x+y, -y)$ is regular also in $I_2'(x, x+y) \cap I_1'(y)$, but, in general, not in a larger domain. This is exactly what is meant by the statement « The envelope of holomorphy of $I_2'(x, y) \cup I_2'(x+y, -y)$ is $I_2'(x, x+y) \cap I_1'(y)$ ».

As explained in ([3]) this result can be obtained also by the technique of ([1]), where the assumption of the equivalence of J.P.C. and commutativity is not assumed, but is proved from (2.8) as follows: Suppose $F(x, y)$ is the regular function in question.

Then

$$ABC(x, y) = \lim_{\varepsilon \to 0} F(x+i\varepsilon,\, y+i\varepsilon)\,; \qquad\qquad \varepsilon \text{ time-like, } \varepsilon_0 > 0;$$

$$ACB(x, y) = \lim_{\varepsilon \to 0} F(x+y+i\varepsilon,\, -y-i\varepsilon)\,; \quad \varepsilon \text{ time-like, } \varepsilon_0 > 0\,.$$

Now if $\mathscr{I}x$ and $\mathscr{I}(x+y) > 0$, y real, the only singularities of the domain $I_2'(x, x+y) \cap I_1'(y)$ are when $y^2 > 0$. Hence if $y \sim 0$ we have $ABC(x, y) = = ACB(x, y)$, $\mathscr{I}x > 0$.

Taking the limit $\mathcal{I}x \to 0$ we get $\langle 0|A[B, C]|0\rangle = 0$, $y \sim 0$ that is, the fields appear to commute. In (^{3}), integral representations were proved for

$$\langle 0|AB \dots C[D, E]|0\rangle$$

which were used to prove that, assuming J.P.C. implies commutativity,

$$(2.9) \qquad EH\{I'_n(x_1, \dots, x_n) \cup I'_n(x_1, \dots, x_{n-1} + x_n, -x_n)\} =$$
$$= I'_n(x_1, \dots, x_{n-1}, x_{n-1} + x_n) \cap I'_1(x_n) .$$

In turn if (2.9) is true, J.P.C. implies local commutativity.

3. – A representation for the triple commutator.

The natural generalization of $\Delta_2(x, y)$ is

$$(3.1) \qquad \Delta_3(x_1, x_2, x_3; m_{ij}) = \int \prod_{\substack{i \leqslant j \\ k}}^{3} \mathrm{d}^4 p_j \exp\left[-ip_k \cdot x_k\right] \delta(p_i \cdot p_j - m_{ij}) \varepsilon(p_{10}) ,$$

a function of 3 four-vectors x_1, x_2, x_3 and six « mass » parameters m_{11}, m_{12}, ..., which sometimes will be collectively denoted by m. This function is zero somewhere in r-space, and has a Fourier transform which is zero somewhere in p-space. As a generalization of (2.5) we are led to consider the following class of functions:

$$(3.2) \qquad T(x, y, z) = \int \varphi(m_{ij}; s, t, \lambda; k) \, \mathrm{d}m \, \mathrm{d}s \, \mathrm{d}t \, \mathrm{d}\lambda \, \mathrm{d}k \, \Delta_3(x, x+y, x+y+z; m_{ij}) \cdot$$
$$\cdot \Delta_2(y, y+z; s, t\,\lambda) \, \Delta_1(z; k) .$$

Theorem. – Functions of the form (3.2) satisfy the following properties (3.3)–(3.6)

$$(3.3) \qquad T(x, y, z) \qquad \text{is Lorentz invariant},$$

$$(3.4) \qquad T(x, y, z) = -T(-x, -y, -z),$$

$$(3.5) \qquad T(x, y, z) = 0 \quad \text{if } z \sim 0 \text{ or if both } y \text{ and } y+z \sim 0,$$

$$(3.6) \qquad \bar{T}(p, q, r) = 0 \quad \text{unless } p, q, r > 0 \text{ or } p, q, r < 0 \text{ or } p, q, q-r > 0$$
$$\text{or } p, q, q-r < 0 \quad \text{or } p, p-q, p-q+r > 0$$
$$\text{or } p, p-q, p-q+r < 0 \quad \text{or } p, p-q, p-r > 0$$
$$\text{or } p, p-q, p-r < 0 .$$

Before proving the theorem we note that these are properties of

$$T(x, y, z) = \langle 0 \,|\, \big[A(x_1), [B(x_2), [C(x_3), D(x_4)]]\big] \,|\, 0 \rangle$$

if $x = x_1 - x_2$, $y = x_2 - x_3$ and $z = x_3 - x_4$.

Proof of (3.3).

Each of the kernels $\Delta_3(x, x+y, x+y+z; m_{ij})$, $\Delta_2(y, y+z; s, t, \lambda)$ and $\Delta(z; k)$ is Lorentz invariant for any value of the parameters, and so the product is Lorentz invariant.

Proof of (3.4).

The kernels are antisymmetric under space-time reversal, for example,

$$\Delta_3(x, y, z) = -\Delta_3(-x, -y, -z)$$

and so the product is, for any value of the parameters.

Proof of (3.5).

The functions $\Delta_3(x, y, z; m_{ij})$, ... are simply related to the « generalized singular functions » $\Delta^{(+)}$ of KÄLLÉN and WILHELMSON ([5]), for example

$$\Delta(x, y, z; m_{ij}) = \Delta^{(+)}(x+i\varepsilon, y+i\varepsilon, z+i\varepsilon; m_{ij}) - \Delta^{(+)}(x-i\varepsilon, y-i\varepsilon, z-i\varepsilon; m_{ij}),$$

where ε is timelike. If $m_{ij} > 0$ then $\overline{\Delta}^{(+)}(p, q, r; m_{ij})$ is non-zero only if $p, q, r > 0$. Hence $\Delta^{(+)}(x, y, z)$ is regular in the forward tube. Hence we can apply the « Edge of the Wedge » theorem ([6]), which says that

$$\Delta^{(+)}(x+i\varepsilon, y+i\varepsilon, z+i\varepsilon; m_{ij}) - \Delta^{(+)}(x-i\varepsilon, y-i\varepsilon, z-i\varepsilon; m_{ij}) = 0 \qquad \text{at } x, y, z$$

if and only if $\Delta^{(+)}(x, y, z)$ is regular at x, y, z, and this for each m_{ij}. So that $T(x, y, z) = 0$ if $z \sim 0$, or if y, z is a real regular point of $\Delta^{(+)}(y, y+z; s, t, \lambda)$ for all « physical » s, t, λ. But the points y, z which lie in the domain of regularity of $\Gamma^{(-)}(y, y+z; s, t, \lambda)$ for all physical s, t, λ make up the extended forward tube ([7]) $I_2'(y, y+z)$ and the real points of $I_2'(y, y+z)$ are known to be Jost points, such that $y + \lambda z \sim 0$ for all λ in the interval $[0, 1]$. We have therefore proved that

$$\Delta(y, y+z; s, t, \lambda) = 0 \qquad \text{for all « physical » } s, t, \lambda$$

if and only if

$$y + \lambda z \sim 0 \qquad\qquad \text{for all } \lambda \text{ in } [0, 1].$$

([5]) E. MONTALDI: *Nuovo Cimento*, **11**, 149 (1959).
([6]) H. J. BREMMERMAN, R. OEHME and J. G. TAYLOR: *Phys. Rev.*, **109**, 2178 (1958).
([7]) Appendix 2 to Ref. ([1]).

We have to show that if y and $y+z \sim 0$, then $T(x, y, z) = 0$. Sufficient for this is $\Delta(y, y+z; s, t, \lambda) = 0$ for all s, t, λ, and so, from what we have just shown, $T(x, y, z) = 0$ if $y + \lambda z \sim 0$ for λ in $[0, 1]$, We may take $z^2 > 0$, since otherwise $T(x, y, z) = 0$ because of the factor $\Delta_k(z)$. Suppose y and $y+z \sim 0$, and the contrary, that there exists λ in $[0, 1]$ such that $(y + \lambda z)^2 > 0$. We will show that this leads to a contradiction. There are 4 cases:

(1) $\qquad z > 0$, $\quad y + \lambda z > 0$. $\qquad$ Then $\quad y + z = (y + \lambda z) + (1 - \lambda) z > 0$

(2) $\qquad z > 0$, $\quad y + \lambda z < 0$. $\qquad$ Then $\qquad y = (y + \lambda z) - \lambda z < 0$

(3) $\qquad z < 0$, $\quad y + \lambda z > 0$. $\qquad$ Then $\qquad y = (y + \lambda z) - \lambda z > 0$

(4) $\qquad z < 0$, $\quad y + \lambda z < 0$. $\qquad$ Then $\quad y + z = (y + \lambda z) + (1 - \lambda) z < 0$.

Here we have used the « Convexity of the Light Cone ». If $x > 0$, $y > 0$, then $\mu x + \lambda y > 0$ for all λ, $\mu > 0$. In each case we obtain a contradiction of the property y and $y+z$ space-like. This proves (3.5).

Proof of (3.6).
We use the definitions of the antisymmetric invariant functions

$$\Delta_1(z; k) = \int \delta(\alpha^2 - k^2)\, \varepsilon(\alpha_0) \exp\left[-i\alpha z\right] d^4\alpha,$$

$$\Delta_2(y, y+z; s, t, \lambda) =$$
$$= \int \delta(\beta_1^2 - s^2)\, \delta(\beta_1\beta_2 - \lambda)\, \delta(\beta_2^2 - t^2)\, \varepsilon(\beta_{10}) \exp\left[-i(\beta_1 y + \beta_2 \overline{y+z})\right] d^4\beta_1\, d^4\beta_2,$$

$$\Delta_3(x, x+y, x+y+z; m_{ij}) =$$
$$= \int \prod_{i<j}^{3} \delta(\gamma_i \cdot \gamma_j - m_{ij})\, \varepsilon(\gamma_{10})\, d^4\gamma_i \exp\left[-i(\gamma_1 x + \gamma_2(x+y) + \gamma_3(x+y+z))\right].$$

Then

$$\overline{T}(p, q, r) = \int \varphi\, dm_{ij}\, ds\, dt\, d\lambda\, dk \int \exp\left[ipx + iqy + irz\right] d^4x\, d^4y\, d^4z \int d^4\alpha\, d^4\beta_i\, d^4\gamma_i \cdot$$
$$\cdot \prod_{i<j}^{3} \delta(\alpha^2 - k^2)\, \varepsilon(\alpha_0)\, \delta(\beta_1^2 - s^2)\, \delta(\beta_1\beta_2 - \lambda)\, \delta(\beta_2^2 - t^2)\, \varepsilon(\beta_{10})\, \delta(\gamma_i\gamma_j - m_{ij})\, \varepsilon(\gamma_{10}) \cdot$$
$$\cdot \exp\left[-i(az + \beta_1 y + \beta_2(y+z) + \gamma_1 x + \gamma_2(x+y) + \gamma_3(x+y+z))\right].$$

The x, y, z integrations give δ'-functions which restrict the integration over the momenta α, β_i, γ_i to the following region

(3.8) $\qquad p = \gamma_1 + \gamma_2 + \gamma_3$; $\qquad q = \gamma_2 + \gamma_3 + \beta_1 + \beta_2$; $\qquad r = \gamma_3 + \beta_2 + \alpha$.

If the parameters m_{ij}, s^2, t^2, λ, k^2 are positive, then the functions

$$\delta(\alpha^2 - k^2)\,\varepsilon(\alpha_0)\,, \quad \delta(\beta_1^2 - s^2)\,\delta(\beta_1\beta_2 - \lambda)\,\delta(\beta_2^2 - t^2)\,\varepsilon(\beta_{10}) \quad \text{and} \quad \prod_{i \leqslant j}^{3} \delta(\gamma_i\gamma_j - m_{ij})\,\varepsilon(\gamma_{10})$$

are non-zero in two separate regions $\alpha > 0$ or $\alpha < 0$; β_1 and $\beta_2 > 0$ or β_1 and $\beta_2 < 0$; γ_1, γ_2 and $\gamma_3 > 0$ or γ_1, γ_2 and $\gamma_3 < 0$ respectively. We have excluded the regions $\alpha \sim 0$, $\beta_i \sim 0$, etc., by choosing k^2, s^2, t^2, $m_{ii} > 0$. The region $\beta_1 > 0$, $\beta_2 < 0$ is excluded because $\beta_1 \cdot \beta_2 = \lambda > 0$. The product

$$\left\{\delta(\alpha^2 - k^2)\,\varepsilon(\sigma_0)\right\}\left\{\delta(\beta_1^2 - s^2)\,\delta(\beta_1 \cdot \beta_2 - \lambda)\,\delta(\beta_2^2 - t^2)\,\varepsilon(\beta_{10})\right\}\left\{\prod_{i \leqslant j}^{3} \delta(\gamma_i \cdot \gamma_j - m_{ii})\,\varepsilon(\gamma_{10})\right\}$$

is therefore zero outside the union of the following sets

$R_{(1)}$	$\alpha > 0$; $\beta_i > 0$; $\gamma_i > 0$.		$R_{(2)}$	$\alpha < 0$; $\beta_i > 0$; $\gamma_i > 0$.
$R_{(3)}$	$\alpha > 0$; $\beta_i < 0$; $\gamma_i > 0$.		$R_{(4)}$	$\alpha < 0$; $\beta_i < 0$; $\gamma_i > 0$.
$R_{(5)}$	$\alpha > 0$; $\beta_i > 0$; $\gamma_i < 0$.		$R_{(6)}$	$\alpha < 0$; $\beta_i > 0$; $\gamma_i < 0$.
$R_{(7)}$	$\alpha > 0$; $\beta_i < 0$; $\gamma_i < 0$.		$R_{(8)}$	$\alpha < 0$; $\beta_i < 0$; $\gamma_i < 0$.

Let

$$S_1 \quad (p,\, q,\, r > 0): \qquad\qquad\qquad S_2 \quad (p,\, q,\, q - r > 0);$$
$$S_3 \equiv (p,\, p - q,\, p - q + r > 0); \qquad S_4 \equiv (p,\, p - q,\, p - r > 0);$$
$$S_5 \equiv (p,\, p - q,\, p - r < 0); \qquad S_6 \equiv (p,\, p - q,\, p - q + r < 0);$$
$$S_7 \equiv (p,\, q,\, q - r < 0); \qquad\qquad S_8 \equiv (p,\, q,\, r < 0).$$

Using (3.8) we now show that if α, β, $\gamma \in R_i$ then p, q, $r \in S_i$.

(1) α, β, $\gamma \in R_1$.   Then $p = \gamma_1 + \gamma_2 + \gamma_3 > 0$; $\quad q = \gamma_2 + \gamma_3 + \beta_1 + \beta_2 > 0$;
$$r = \gamma_3 + \beta_2 + \alpha > 0\,.$$

Hence p, q, $r \in S_1$.

(2) α, β, $\gamma \in R_2$.   Then $q - r = \beta_1 + \gamma_2 - \alpha > 0$;   so p, q, $r \in S_2$.

(3) α, β, $\gamma \in R_3$.   Then $p > 0$; $\quad p - q = \gamma_1 - \beta_1 - \beta_2 > 0$;
$$p - q + r = \gamma_1 + \gamma_3 - \beta_1 + \alpha > 0; \quad \text{so} \quad p,\, q,\, r \in S_3\,.$$

(4) α, β, $\gamma \in R_4$.   Then $p > 0$, $\quad p - q > 0$ as in (3);
$$p - r = \gamma_1 + \gamma_2 - \beta_2 - \alpha > 0\,. \quad \text{Hence} \quad p,\, q,\, r \in S_4\,.$$

Similarly if $\gamma_i < 0$ we can prove that if $\alpha, \beta, \gamma \in R_i$ then $p, q, r \in S_i$, $i = 5, ..., 8$. This proves that the spectrum of $T(x, y, z)$ is $\bigcup_{i=1}^{8} S_i$. since the α, β, γ vary over $\bigcup_{i=1}^{8} R_i$ subject to (3.8). This proves (3.6).

We now conjecture the converse to this theorem, namely, that *every* function with the properties (3.3)–(3.6) has the representation (3.2). This is a reasonable conjecture in view of (2.5) and the theorem which precedes it. Functions of the form (3.2) also vanish in a larger region than (3.5). Because of the factor $\Delta(x, x+y, x+y+z; m_{ij})$, $T = 0$ if $x, x+y, x+y+z$ are J.P. But functions of the form (3.2) do not in general ([2]) vanish if $x, x+y$ and $x-y-z$ are space-like, which is a property of the triple commutator. Also, these functions do not satisfy the algebraic relations corresponding to the Jacobi identities. If (2.9) is true, the incorporation of the Jacobi identities would automatically ensure that T satisfied the further property: $T(x, y, z) = 0$ if $x, x+y$ and $x+y+z$ are spacelike.

Unlike the double commutator, the triple commutator has such a complicated overlap of spectra that the various terms $ABCD, ABDC, ...,$ cannot be separated cleanly, and so that even if (3.2) is the most general form we do not obtain immediately the envelope of holomorphy of the 4 extended tubes of $ABCD, ABDC, ACDB, ADCB$. To do this we would have to find the *most general* form for a function regular in

$$T = I_3'(x, y, z) \cup I_3'(x, y+z, -z) \cup I_3'(x+y, z, -y-z) \cup I_3'(x+y+z, -y, -z) .$$

We can construct examples of functions regular in T. Consider.

$$(3.9) \qquad f(x, y, z) =$$

$$= \int \varphi(m_{ij}; s, t, \lambda; k) \Delta_3^{(+)}(x, x+y, x+y+z; m_{ij}) \Delta_2^{(+)}(y, y+z; s, t, \lambda) \Delta^{(+)}(z, k) .$$

The function (3.9) is the Fourier transform of a function like (3.7), but with $\varepsilon(\alpha_0) \varepsilon(\beta_{10}) \varepsilon(\gamma_{10})$ replaced by $\theta(\gamma_{10}) \theta(\pm \alpha_0) \theta(\pm \beta_{10})$ the sign $\pm$ depending on the sign of the imaginary parts of x, y, z. As was shown in the proof of (3.6), taking one of the signs $\pm \alpha_0$, $\pm \beta_{10}$ restricts $\bar{f}(p, q, r)$ to lie in one of the sets $p, q, r > 0$; $p, q, q-r > 0$; $p, p-q, p-q+r > 0$; $p, p-q, p-r > 0$; and so (3.9) is regular in a tube conjugate to one of these. So we have constructed four functions, regular respectively in $I_3'(x, y, z)$, $I_3'(x, y+z, -z)$, $I_3'(x+y, z, -y-z)$ and $I_3'(x+y+z, -y, -z)$. The difference between, for example, the first two is the Fourier transform of (3.7) with $\varepsilon(\gamma_{10}) \theta(\beta_0) \varepsilon(\alpha_0)$ in place of $\varepsilon(\gamma_{10}) \varepsilon(\beta_{10}) \varepsilon(\alpha_0)$, and this is zero somewhere in x-space. Hence the functions continue one another. This proves that (3.9) is regular in T. The singula-

rities of (3.9) lie on the boundary of

$$(3.10) \qquad I_3'(x,\ x+y,\ x+y+z) \cap I_2'(y,\ y+z) \cap I_1'(z)\ .$$

We have proved that in general a function regular in T can have singularities on the boundary of (3.10), and so the envelope of holomorphy of T is contained in (3.10). By analogy with (2.8) we can conjecture that (3.10) is the actual EH of T, $i.e.$

$$(3.11) \quad EH\left\{I_3'(x,\ y,\ z) \cup I_3'(x,\ y+z,\ -z) \cup I_3'(x+y,\ z,\ -y-z) \cup \right.$$
$$\left. \cup\ I_3'((x+y+z),\ -z,\ -y)\right\} = I_3'(x,\ x+y,\ x+y+z) \cap I_2'(y,\ y+z) \cap I_1'(z)\ .$$

Eq. (3.11) is a useful « Ansatz » with which to start the analysis of the 4-fold expectation values.

4. – Other integral representations.

The natural generalizations of the representations for $D(x, y)$ and $T(x, y, z)$ would be

$$(4.1) \qquad F_n(y_1, ..., y_{n-1}) = \int \varphi(m_{ij};\ ...;\ k)\,\mathrm{d}m_{ij}\ ...\ \mathrm{d}k\ \Delta(y_{n-1};\ k)\ ...$$
$$... \Delta_{n-1}(y_1,\ y_1+y_2,\ y_1+y_2+y_3,\ ...;\ m_{i:})\ ,$$

where

$$(4.2) \qquad F(y_1, ..., y_{n-1}) = \langle 0\,|\,\big[A(x_1),\ [B(x_2),\ ...\ [D(x_{n-1}),\ E(x_n)]\ ...]\big]\,|\,0\rangle\ .$$

It is a remarkable fact, proved in ([4]), that (4.1) has the spectrum of the multiple commutator (4.2). For n 4 this was proved in the previous section. The representation (4.1) does not satisfy all the consequences of causality, because it is not zero in x-space in as large a region as is the right hand side of (4.2). But (4.1) does satisfy the consequences of J.P.C., and would satisfy causality if the Jacobi identities could be incorporated. It is important to note that these results are merely *algebraic*, and that the analytic problem of showing that these are the *most general* has not been solved. In the same way as (3.11) was derived we can get a conjecture for the envelope of holomorphy of 2^{n-2} extended tubes at once in the n-fold case, the tubes being those which occur in the multiple commutator. In ([3]) it was shown that in the 3-fold case, by dealing with the three functions in pairs, and then combining the results, we can follow through the working of ([1]) up to a certain

point. To finish the problem by this method would require the Jacobi identities. In the four fold case we can deal with only 4 out of 12 at once, and these must be the special 4 occurring in a triple commutator. The effect of the Jacobi identity is therefore more important.

Spectral representations for a number of other forms for the commutator brackets were proposed in ([1])

$$(4.3) \quad \langle 0 \, | \, [[A, B], [C, D]] \, | \, 0 \rangle = \int \varphi(k, m, n_j) \, \Delta_4(y+z, y, x+y+z, x+y; n_j) \cdot$$
$$\cdot \, l(x; k) \, \Delta(z; m) \, dm \, dn_j \, dk \,,$$

$$(4.4) \quad \langle 0 \, | \, [[A, B], [C \, [D, E]]] \, | \, 0 \rangle = \int \varphi \, \Delta(x; l) \, \Delta(w, m) \, \Delta_2(z, z+w; n) \, dl \, dm \, dn \cdot$$
$$\cdot \, \Delta_6(x+y, x+y+z, x+y+z+w, y, y+z, y+z+w) \,.$$

Here A has argument x_1, B has argument x_2, etc., and $w = x_4 - x_5$. In ([1]) we also proposed representations (not proved the most general) for the difference of any two 4-fold functions. As one example

$$(4.5) \quad \langle ABCD \rangle - \langle BDCA \rangle = P(x^2) \int \varphi(m, n) \, \Delta_4(x, x+y, z, x+\dot{y}+z; n) \cdot$$
$$\cdot \, \Delta_2^{(+)}(y, y+z; m) \, dm \, dn \,,$$

where $P(x^2)$ is some polynomial. Then consider the two functions

$$ABCD = P(x^2) \int \varphi \, dm \, dn \, \Delta_2^{(+)}(y, y+z; m) \cdot$$
$$\cdot \, \Delta_4^{(+)}(x+i\varepsilon, x+y+i\varepsilon, z+i\varepsilon, x+y+z+i\varepsilon; n) \,,$$

$$BDCA = P(x^2) \int \varphi \, dm \, dn \, \Delta_2^{(+)}(y, y+z; m) \cdot$$
$$\cdot \, \Delta_4^{(+)}(x-i\varepsilon, x+y-i\varepsilon, z-i\varepsilon, x+y+z-i\varepsilon; n) \,.$$

These are regular respectively in the extended tubes $I_3'(x, y, z)$ and $I_3'(x+y, -y-z, z)$, and since they are equal *somewhere* in x-space, they continue one another into the union

$$(4.6) \quad I_3'(x, y, z) \cup I_3'(x+y, -y-z, z) \,.$$

But it is known that the singularities of $\Delta_2^{(+)}(y, y+z; m)$ and $\Delta_4^{(+)}(x, x+y, z, x+y+z; n)$ as m, n vary over the physical region, lie outside

the domain $I_2'(y, y+z)$ and $I_4'(x, x+y, z, x+y+z)$ and so (4.5) defines functions regular in

(4.7) $$I_4'(x, x+y, z, x+y+z) \cap I_2'(y, y+z) .$$

Therefore

(4.8) $EH\{I_3'(x, y, z) \cup I_3'(x+y, -y-z, z)\} \subset$

$$\subset I_4'(x, x+y, z, x+y+z) \cap I_2'(y, y+z) .$$

RIASSUNTO (*)

Si mostra che le proprietà di massa ed energia positiva e di causalità portano nella teoria quantistica dei campi a problemi di teoria del completamento analitico. Si formula una rappresentazione del doppio commutatore che precedentemente si è dimostrata la più generale. Si dimostra che una rappresentazione analoga per il commutatore triplo soddisfa alle condizioni di causalità e di massa ad energia positiva, ma non mostra di dare tutte le funzioni con queste proprietà. Si fa una congettura sugli inviluppi olomorfici dei domini che si incontrano nella discussione delle proprietà analitiche della funzione di Green dei quattro punti. Si propongono rappresentazioni integrali per il commutatore n-plo, e per alcune altre combinazioni delle funzioni di Wightman. Con queste rappresentazioni si ottengono alcuni risultati riguardanti il completamento analitico.

The Gauge Properties of Green's Functions
in Quantum Electrodynamics (*).

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Summary. — Effects of a change of gauge on the Green's functions are studied systematically in quantum electrodynamics and relations between Green's functions under two different gauges are given.

Recently, some interest has been expressed concerning the gauge properties of Green's functions. Several authors [1-3] have derived formulae which give the change of Green's functions under a change of the gauge. However, their formulae are restricted to a certain type of simple Green's functions.

The purpose of this note is to investigate this problem systematically and to derive a generalization of their results for general Green's functions. In this paper, we use the method which has been obtained by CAIANIELLO [4]. (Hereafter, we refer this as II.)

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[3] H. M. FRIED: *Phys. Rev.*, **115**, 220 (1959).
[4] E. R. CAIANIELLO: *Nuovo Cimento*, **11** 220 (1954).

According to eq. (30) of II, the general Green's function with $2N$ external electrons and P photons lines is defined by

$$(1) \qquad K_{\mu_1 \ldots \mu_p}\begin{pmatrix} x_1 \ldots x_N \\ y_1 \ldots y_N \end{pmatrix} \Big| t_1 \ldots t_p \Big) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int\int \ldots \int d\xi_1 \ldots d\xi_n \cdot$$

$$\cdot \gamma_1 \ldots \gamma_n \begin{pmatrix} x_1 \ldots x_N \ \xi_1 \ldots \xi_n \\ y_1 \ldots y_N \ \xi_1 \ldots \xi_n \end{pmatrix} [t_1 \ldots t_p \ \xi_1 \ldots \xi_n]_{\mu_1 \ldots \mu_p},$$

where $\mu_1 \ldots \mu_p$ represent the vector indices for the external photon lines and where we have suppressed all other indices such as the spinor ones for simplicity. Otherwise, all notations are the same as in II.

To begin with, we prove the following

Lemma.

$$(2) \qquad \left(\gamma_\mu \frac{\hat{\partial}}{\partial \xi_\mu}\right) \cdot K \begin{pmatrix} \xi \ x_1 \ldots x_N \\ \xi \ y_1 \ldots y_N \end{pmatrix} t_1 \ldots t_p \Big)$$

$$i \cdot \sum_{j=1}^{N} [\delta(x_j - \xi) - \delta(y_j \quad \xi)] \cdot K \begin{pmatrix} x_1 \ldots x_N \\ y_1 \ldots y_N \end{pmatrix} \Big| t_1 \ldots t_p \Big),$$

where we have also omitted the photon indices for simplicity, and it is understood that we take a contraction of the γ-matrix on the left hand side of eq. (2) with the spinor indices which are contained in two ξ inside the bracket of the Green's function.

Eq. (2) is essentially equivalent to the conservation of the electric current in quantum electrodynamics. This can be easily proved, if we note the following identities.

$$\left(\gamma \frac{\partial}{\partial \xi} + m\right) (\xi, y) = - i \, \delta(\xi - y),$$

$$(x, \xi) \cdot \left(\gamma \frac{\hat{\partial}}{\partial \xi} - m\right) \quad - i \, \delta(\xi - x).$$

Then, eq. (2) immediately follows from eq. (1).

Now, in what follows, we use the notation $D_{\mu\nu}(t, t')$ for the causal propagator of the free photon field instead of $[t, t']$ of II. Actually, $D_{\mu\nu}(t, t')$ is not well defined due to the ambiguous contribution from the scalar photons. Generally, $D_{\mu\nu}$ has the following form:

$$(3) \qquad D_{\mu\nu}(t, t') = D_{\mu\nu}^{(0)}(t, t') + \frac{\hat{\partial}^2}{\partial t_\mu \, \partial t'_\nu} f(t, t'),$$

where $D_{\mu\nu}^{(0)}(t, t')$ is the value of $D_{\mu\nu}(t, t')$ at a fixed gauge and f is an arbitrary function except for a symmetry condition of interchange of t and t'. Actually from the translation invariance $f(t, t')$ must have the form $g(t - t')$ but in what follows this is completely unnecessary so that we can include a possibility of a non-translation invariant gauge (*).

Now, our problem is to investigate the transformation properties of the Green's functions under a change of the gauge according to eq. (3). More precisely, we want to know the f-dependence of the Green's functions. First, we will prove the following theorem.

Theorem 1. – Under the transformation eq. (3), we have

$$
(4) \qquad K \begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix} = \exp\left[-\lambda^2 \cdot I\right] \cdot K^{(0)} \begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix},
$$

where

$$
(5) \qquad I = \tfrac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[f(x_i, x_j) + f(y_i, y_j) - f(x_i, y_j) - f(y_i, x_j) \right]
$$

and $K^{(0)}$ is the value of K in a fixed gauge, namely the value of K when we replace $D_{\mu\nu}$ by $D_{\mu\nu}^{(0)}$.

Note that for the special case $N = 1$ and when $f(t_1, t_2) \equiv f(t_1 - t_2)$ this gives

$$
(6) \qquad K \begin{pmatrix} x \\ y \end{pmatrix} = \exp\left[\lambda^2 (f(x - y) - f(0)) \right] \cdot K^{(0)} \begin{pmatrix} x \\ y \end{pmatrix}.
$$

This is the result proved recently by ZUMINO and JOHNSON and SCWHINGER (²). Eq. (4) is a generalization of their result. Now, we prove eq. (4). Let us use the formula eq. (55) of II:

$$
(7) \qquad \frac{\mathrm{d}}{\mathrm{d}(\lambda^2/2)} K \begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix} = \int \mathrm{d}\xi_1 \int \mathrm{d}\xi_2 \, \gamma_{\mu_1} \gamma_{\mu_2} D_{\mu_1 \mu_2}(\xi_1, \xi_2) \cdot K \begin{pmatrix} x_1 \dots x_N \ \xi_1 \xi_2 \\ y_1 \dots y_N \ \xi_1 \xi_2 \end{pmatrix}.
$$

Again, we have omitted spinor indices. This does not lead to any confusion, if we are careful.

Now, insert eq. (3) into the right hand side of eq. (7). For parts containing the derivative of f, we integrate them by parts with respect to ξ_1 and ξ_2 and

(*) *Note added in proof.* – Roughly speaking, this corresponds to a *c*-number gauge transformation, instead of the *q*-number one.

use our lemma eq. (2). Then, it is easy to see that eq. (7) gives

$$
\frac{\mathrm{d}}{\mathrm{d}(\lambda^2/2)} K \begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix} =
$$
$$
= \int\!\!\int \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \, \gamma_{\mu_1}\gamma_{\mu_2} D^{(0)}_{\mu_1,\mu_2}(\xi_1, \xi_2) \cdot K \begin{pmatrix} x_1 \dots x_N \ \xi_1\xi_2 \\ y_1 \dots y_N \ \xi_1\xi_2 \end{pmatrix} - 2I \cdot K \begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix},
$$

where I is given by eq. (5).

So, if we define F by

$$
(8) \qquad F \begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix} = \exp\left[\lambda^2 \cdot I\right] \cdot K \begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix},
$$

F satisfies the following equation

$$
(9) \qquad \frac{\mathrm{d}}{\mathrm{d}(\lambda^2/2)} F \begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix} = \int\!\!\int \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \, \gamma_{\mu_1}\gamma_{\mu_2} D^{(0)}_{\mu_1,\mu_2}(\xi_1, \xi_2) \, F \begin{pmatrix} x_1 \dots x_N \ \xi_1\xi_2 \\ y_1 \dots y_N \ \xi_1\xi_2 \end{pmatrix}.
$$

This is exactly the same as eq. (7), if we replace D by $D^{(0)}$. Furthermore, for $\lambda^2 = 0$, both F and $K^{(0)}$ agree with each other. So we conclude that $F = K^{(0)}$ and this gives our desired result eq. (4) due to eq. (8). The uniqueness of solutions of eq. (7) or (9) with given values of F or K at $\lambda^2 = 0$ can be easily seen as follows. Take the derivatives of both sides of eq. (9) n-times with respect to $\lambda^2/2$, put $\lambda^2 = 0$, then use the Taylor expansion of F into a power series of λ^2. This gives, after some algebra, the desired formula eq. (1).

It is worth-while to note that a similar but more complicated situation has been already investigated by CAIANIELLO [5] in the renormalization problem of the quantum electrodynamics. Also, for a treatment of the infra-red divergence, a similar technique has been applied and will be published [6] shortly.

At this stage, we would like to make several comments. First of all, $K \begin{pmatrix} \xi_1 \dots \xi_n \\ \xi_1 \dots \xi_n \end{pmatrix}$ is independent of any gauge variable f, as we can easily check from eq. (4). This remark will be utilized shortly. Secondly, eq. (4) holds not only for quantum electrodynamics but also for a neutral vector meson theory and a neutral scalar meson theory with vector coupling. For example, if we use the Stückelberg formulation [7] for the vector field, the function f

[5] E. R. CAIANIELLO: *Nuovo Cimento*, **13**, 637 (1959); **14**, 185 (1959).

[6] S. OKUBO and E. R. CAIANIELLO: to be published.

[7] E. C. G. STÜCKELBERG: *Helv. Phys. Acta*, **11**, 225 (1938).

in eq. (3) is essentially the contribution from the auxiliary scalar field and is given by $(1/\varkappa^2)\varDelta_F$. Then, formula eq. (4) represents the usual statement [8] that after a unitary transformation, the scalar field disappears and the effective Green's function is given by $K^{(0)}$, thus making the theory renormalizable in this case. For the vector coupling theory of a neutral scalar meson, the situation is much simpler. In this case, we have only to put in eq. (3)

$$D_{\mu\nu}^{(0)}(x, y) \equiv 0 , \qquad f(x, y) = \varDelta_F(x - y)$$

so that

$$K \begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix} = \begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix} \cdot$$
$$\cdot \exp \left[-\frac{\lambda^2}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} [\varDelta_F(x_i - x_j) + \varDelta_F(y_i - y_j) - \varDelta_F(x_i - y_j) - \varDelta_F(y_i - x_j)] \right] .$$

Special cases of $N = 1$ and 2 of this formula have been sometime ago given by the author [9] by using a different method.

We note that all formulae to be given afterwards apply also for both the vector meson and scalar meson theories as well as for quantum electrodynamics.

Returning to our original problem, let us now define the photon propagator $g_{\mu\nu}$ by

$$g_{\mu\nu}(t, t') = K_{\mu\nu}(t, t')/K_0 ,$$

where K_0 is the contribution from the vacuum fluctuation for the kernel. Then, we can prove the following

Theorem 2.

(10)
$$g_{\mu\nu}(t, t') = g_{\mu\nu}^{(0)}(t, t') + \frac{\widehat{c}^2}{\partial t_\mu \partial t_\nu'} f(t, t').$$

Eq. (10) implies that the gauge-dependent term is unaffected by the interaction, which has also been proved by several authors [1 2]. We use eq. (48) of II, iterating it twice

$$K_{\mu\nu}(t, t') = D_{\mu\nu}(t, t') K_0 + \lambda^2 \iint d\xi_1 d\xi_2 D_{\mu x}(t, \xi_1)(\gamma_1)_x D_{\nu\beta}(t', \xi_2)(\gamma_2)_\beta K \begin{pmatrix} \xi_1 \xi_2 \\ \xi_1 \xi_2 \end{pmatrix} .$$

In exactly the same manner, we insert eq. (3) into the D-functions inside the integrand of the above equation and make partial integrations for the

[8] R. J. GLAUBER: *Prog. Theor. Phys. (Japan)*, **9**, 295 (1953).
[9] S. OKUBO: *Prog. Theor. Phys. (Japan)*, **11**, 80 (1954).

parts containing the gauge term f. Again, utilizing our lemma eq. (2), we find that these do not contribute at all. So, we have

$$(11) \qquad K_{\mu\nu}(t,\,t') \quad D_{\mu\nu}(t,\,t')\,K_0 - \lambda^2 \iint d\xi_1\,d\xi_2\,D_{\mu\gamma}^{(0)}(t,\,\xi_1)(\gamma_1)_\gamma\,D_{\nu\beta}^{(0)}(t',\,\xi_2)(\gamma_2)_\beta\,K\begin{pmatrix}\xi_1\,\xi_2\\\xi_1\,\xi_2\end{pmatrix}.$$

Furthermore, if we remember the remark made just after the proof of the Theorem 1 that $K\begin{pmatrix}\xi_1\,\xi_2\\\xi_1\,\xi_2\end{pmatrix}$ and K_0 are independent of the gauge, then eq. (11) gives immediately eq. (10).

Theorem 3.

$$(12) \qquad \frac{\partial}{\partial t_\mu}\,g_{\mu\nu}(t,\,t') \equiv \frac{\partial}{\partial t_\mu}\,D_{\mu\nu}(t,\,t')\,.$$

This can be easily derived from eq. (11). We may choose from the beginning the fixed gauge so that

$$(13) \qquad \frac{\partial}{\partial t_\mu}\,D_{\mu\nu}^{(0)}(t,\,t') = 0\,.$$

Then, we get eq. (12) from eq. (11). This has been also recently proved by FRIED ([3]).

Now, we will prove the most general case, a generalization of Theorem 1 and 2 for arbitrary Green's functions. To avoid unnecessary complications, we define the following quantities

$$(14) \qquad \begin{cases} g_h \equiv -\,i\lambda\cdot\displaystyle\sum_{j=1}^{N}\frac{\partial}{\partial (t_h)_\mu}\,[f(t_h,\,x_j) - f(t_h,\,y_j)]\,,\\[2mm] f_{h,k} \equiv \dfrac{\partial^2}{\partial t_{\mu_h}\,\partial t_{\mu_k}}\,f(t_h,\,t_k)\,. \end{cases}$$

Theorem 4.

$$(15) \qquad \exp\,[\lambda^2\cdot I]\cdot K\begin{pmatrix}x_1\dots x_N\\y_1\dots y_N\end{pmatrix}t_1\dots t_p\Big) = K^{(0)}\begin{pmatrix}x_1\dots x_N\\y_1\dots y_N\end{pmatrix}t_1\dots t_p\Big) +$$

$$+\sum_{h=1}^{p}g_h\cdot K^{(0)}\begin{pmatrix}x_1\dots x_N\\y_1\dots y_N\end{pmatrix}t_1\dots t_p\Big)_{t_j} +$$

$$+\sum_{(h\neq k)}^{p}\sum^{p}[g_h\cdot g_k + f_{h,k}]\,K^{(0)}\begin{pmatrix}x_1\dots x_N\\y_1\dots y_N\end{pmatrix}t_1\dots t_p\Big)_{t_h,t_k} +$$

$$+\sum_{(h\neq k\neq l\neq h)}^{p}\sum^{p}\sum^{p}[g_h g_k g_l + g_h f_{k,l} + g_k f_{l,h} + g_l f_{h,k}]\,K^{(0)}\begin{pmatrix}x_1\dots x_N\\y_1\dots y_N\end{pmatrix}t_1\dots t_p\Big)_{t_h,t_k,t_l} + \dots,$$

where

$$K^{(0)} \begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix} t_1 \dots t_p \Big)_{t_\alpha t_\beta t_\gamma \dots}$$

means to omit the arguments t_α, t_β, t_γ, ... from $t_1 \dots t_p$ inside the bracket, $e.g.$

$$K^{(0)} \begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix} t_1 \dots t_p \Big)_{t_h} = K^{(0)} \begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix} t_1 \dots t_{h-1} t_{h+1} \dots t_p \Big).$$

We sketch the proof of eq. (15). Use eq. (48) of II,

$$(16) \qquad K \begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix} t_{p+1} t_1 \dots t_p \Big) = \sum_{h=1}^{p} D_{\mu_{p+1}, \mu_h}(t_{p+1}, t_h) \, K \begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix} t_1 \dots t_p \Big)_{t_h} +$$

$$+ \lambda \int d\xi \, (\gamma)_\nu \, D_{\mu_{p+1}, \nu}(t_{p+1}, \xi) \, K \begin{pmatrix} x_1 \dots x_N \, \xi \\ y_1 \dots y_N \, \xi \end{pmatrix} t_1 \dots t_p \Big).$$

Again, insert eq. (3) into the right hand side of eq. (16) and integrate in parts the integral containing f. Then, eq. (16) goes to

$$(17) \qquad K \begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix} t_{p+1} t_1 \dots t_p \Big) = \sum_{h=1}^{p} D^{(0)}_{\mu_{p+1}, \mu_h}(t_{p+1}, t_h) \, K \begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix} t_1 \dots t_p \Big)_{t_h} +$$

$$+ \lambda \int d\xi(\gamma)_\nu \, D^{0}_{\mu_{p+1}, \nu}(t_{p+1}, \xi) \, K \begin{pmatrix} x_1 \dots x_N \, \xi \\ y_1 \dots y_N \, \xi \end{pmatrix} t_1 \dots t_p \Big) -$$

$$- g_{p+1} \cdot K \begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix} t_1 \dots t_p \Big) + \sum_{h=1}^{p} f_{p+1, h} \cdot K \begin{pmatrix} x_1 \dots x_N \\ y_1 \dots y_N \end{pmatrix} t_1 \dots t_p \Big)_{t_h}.$$

We proceed by mathematical induction. Suppose that the formula eq. (15) is true for $p = p_0$, then insert it into the right hand side of eq. (17) and use eq. (16) in a fixed gauge. Then, eq. (17) assures us that eq. (15) is also true for $p = p_0 + 1$. Now for $p_0 = 0$, eq. (15) holds because of our Theorem 1 (eq. (4)). Thus, eq. (15) holds generally. Theorem 4 covers our theorem 1 and 2 as special cases.

As an application of our Theorem 4, we now derive a generalization of Ward's identity ([1,10]). In momentum space

$$(18) \qquad - i(p - q)_\nu \cdot S_F'(p) \Gamma_\nu(p, q) S_F'(q) = S_F'(p) - S_F'(q).$$

This can be obtained by taking the Fourier transform of the following

equation and using Theorem 3, as was shown by TAKAHASHI [10]

$$(19) \qquad \Box_y \cdot \frac{\partial}{\partial y_\mu} K_\mu \begin{pmatrix} x \\ x' \end{pmatrix} y = \lambda \left[\delta(y - x') - \delta(y - x) \right] \cdot K \begin{pmatrix} x \\ x' \end{pmatrix} .$$

TAKAHASI [10] proved this by using the vacuum expectation value of Heisenberg operators. However, this method is rather questionable, whenever there are difficulties with gauge invariances. Actually, we will prove that eq. (19) is not true in general but holds only in a special gauge, the usual Feynman gauge; *i.e.*

$$(20) \qquad D_{\mu\nu}(x, y) = \frac{-i}{(2\pi)^4} \int \mathrm{d}^4 k \frac{\exp[ik(x-y)]}{k^2 - i\varepsilon} \cdot \delta_{\mu\nu} .$$

By our Theorem 4, (eq. (15)) and using definitions eq. (14), we have [11]

$$(21) \qquad K_\mu \begin{pmatrix} x \\ x' \end{pmatrix} y \equiv \left[K_\mu^{(0)} \begin{pmatrix} x \\ x' \end{pmatrix} y - i\lambda \frac{\partial}{\partial y_\mu} (f(y-x) - f(y-x')) \cdot K^{(0)} \begin{pmatrix} x \\ x' \end{pmatrix} \right] \cdot$$
$$\cdot \exp[\lambda^2(f(x-x') - f(0))] ,$$

when we put

$$f(x, y) \equiv f(x - y) .$$

Note that by eq. (48) of II,

$$(22) \qquad K_\mu^{(0)} \begin{pmatrix} x \\ x' \end{pmatrix} y = \lambda \int \mathrm{d}\xi \, D_{\mu\nu}^{(0)}(y, \xi) \gamma_\nu \, K^{(0)} \begin{pmatrix} x\ \xi \\ x'\xi \end{pmatrix} .$$

Now, from the beginning, we can choose the fixed gauge $D_{\mu\nu}^{(0)}$ which satisfies eq. (13) namely

$$\frac{\partial}{\partial t_\mu} D_{\mu\nu}^{(0)}(t, t') = 0 .$$

Then, from eq. (22) it follows that

$$\frac{\partial}{\partial y_\mu} K_\mu^{(0)} \begin{pmatrix} x \\ x' \end{pmatrix} y \equiv 0 .$$

[10] Y. TAKAHASHI: *Nuovo Cimento*, **6**, 371 (1957).
[11] See also, E. R. FRADKIN: *Žurn. Eksp. Teor. Fiz. U.S.S.R.*, **29**, 258 (1955) (*Soviet Physics, J.E.T.P.*, **2**, 361 (1956)); B. ZUMINO: reference [2].

Thus, eq. (21) gives

$$(23) \qquad \Box_y \cdot \frac{\partial}{\partial y_\mu} K_\mu \left(\begin{matrix} x \\ x' \end{matrix} \middle| y \right) = - i \lambda (\Box_y)^2 [f(y-x) - f(y-x')] \cdot K \left(\begin{matrix} x \\ x' \end{matrix} \right).$$

This obviously differs from eq. (19). However, if we use the gauge eq. (20), then we must have

$$f(x-y) = \frac{-i}{(2\pi)^4} \int d^4 k \, \frac{\exp [ik(x-y)]}{(k^2)^2} \, ,$$

$$D^{(0)}_{\mu\nu}(x-y) = \frac{-i}{(2\pi)^4} \int d^4 k \, \frac{\exp [ik(x-y)]}{k^2} \cdot \left[\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right],$$

so that

$$(\Box_y)^2 \cdot f(x-y) = - i \, \delta(x-y) \, .$$

Then this gives eq. (19).

Finally, we would like to comment that scattering matrix elements are independent of gauge, though Green's functions are not. To see this, we note that we must restrict ourselves for the arbitrary gauge function f by (*)

$$(24) \qquad \lim_{x_0 \to \pm \infty} f(x, y) = \lim_{y_0 \to \pm \infty} f(x, y) = 0 \, .$$

This condition is actually necessary, since we omitted all the partially integrated parts in our previous derivations for our formulae. To derive S-matrix-elements on the energy shell from the Green's functions, we must make the following operations for them ([4, 12])

$$L^{(-)} = \lim_{T \to \infty} \frac{1}{T} \int_{-2T}^{-T} dx_0 \, , \qquad L^{(+)} = \lim_{T \to \infty} \frac{1}{T} \int_{T}^{2T} dx_0 \, .$$

So, we can see that the arbitrary gauge function f does not give any contributions under this operation due to the condition eq. (24) (e.g. see eq. (5))

The factor containing $f(0)$ does not matter, since it gives the renormalization coefficient.

(*) Note added in proof. – Actually this condition seems to be a little restrictive for practical problems. It may be sufficient to impose $\lim_{x_0 \to \pm \infty} f(x, y)$ constant independent·fy, in order to secure the equivalence of the matrix element, apart from the amplitude renormalization.

([12]) M. Gell-Mann and F. Low: Phys. Rev., 84, 350 (1951).

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RIASSUNTO (*)

Si studiano sistematicamente in elettrodinamica quantistica gli effetti di un cambiamento di gauge sulle funzioni di Green e si danno le relazioni fra le funzioni di Green nel caso di due gauge diversi.

(*) *Traduzione a cura della Redazione*

Canonical Transformation and Perturbation Expansion in the Theory of Fermi Gas (*).

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Summary. — A canonical transformation, identical with that of Bogoljubov and Valatin, is performed; and a perturbation expansion of the ground state energy is made by taking, as the unperturbed Hamiltonian, the term describing free « quasi particles ». The perturbation is written in normal form: it follows that the canonical transformation cancels, to all orders, diagrams containing lines with the two end points at the same vertex. A simple discussion of the ground state energy is given in the case considered by the theory of Bardeen, Cooper and Schrieffer.

1. – Introduction.

The mathematical aspects of the new theory of superconductivity of Bardeen, Cooper and Schrieffer [1] have been clarified indipendently by Bogoljubov [2,3] and Valatin [4], using a canonical transformation introduced for the first time by Bogoljubov in 1947 for a Bose system. A short but very clear review of the method can be found in the lectures of Beliaev [5].

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(1) J. Bardeen, L. Cooper and J. R. Schrieffer: *Phys. Rev.*, **108**, 1175 (1957).

(2) N. N. Bogoljubov: *Nuovo Cimento*, **7**, 794 (1958).

(3) N. N. Bogoljubov, V. V. Tolmachev and D. V. Shirkov: *A New Method in the Theory of Superconductivity* (Moscow, 1958).

(4) J. G. Valatin: *Nuovo Cimento*, **7**, 843 (1958).

(5) S. T. Beliaev: *Introduction to the Bogoljubov Canonical Transformation Method.* Cours données a l'école d'été de physique théorique (Les Houches, Session 1958): *Le problème a N corps* (New York, 1958).

The equation which determines the canonical transformation (eq. (10)) has been obtained by BOGOLJUBOV using the principle of compensation of dangerous graphs, and the same equation has been obtained by VALATIN minimizing the expectation value of H in the trial ground state of BARDEEN, COOPER and SCHRIEFFER.

TOLMACHEV and TIABLIKOV ([6]) have extended the method of the compensation of the « dangerous graphs » to second order in the perturbation theory by splitting the Hamiltonian into the parts $\varepsilon_0 + H_0$ and $H_1 + H_2$ (according to their notations) ([*]).

We believe that a more useful splitting of the Hamiltonian can be obtained by taking, as the unperturbed part, the term describing free « quasi particles ». By doing so, we have the following formal advantages:

   a) No dangerous denominators can appear to any order of perturbation theory if there is an energy gap in the excitation spectrum of the « quasi particles ».

   b) The equation (10) which determines the canonical transformation remains unchanged.

   c) The « perturbation » can be written in a more compact form.

We have performed a perturbation expansion in order to find the first non-vanishing corrections to the ground state energy; finally we have applied our formulae to the simple case considered by the new theory of superconductivity, i.e. with the interaction between pairs of zero total momentum effective only in a small shell on the surface of the Fermi sphere.

Let us now devote a few comments to the point a). As has been proved by VAN HOVE ([6a]) the usual Goldstone-Hugenholtz perturbation theory has a radius of convergence zero in the coupling constant of the interaction, if the singlet scattering length of two body potential is negative; and this is due precisely to the « dangerous » denominators. But in our formulation of the theory all the energy denominators are greater than a multiple of the minimum value of $E(k)$ (see eq. (14), (16) of this paper), i.e. are greater than a multiple of the gap of the excitation spectrum of the quasi particles.

Now $E(k)$ is given by eq. (16) and therefore the condition $E(k) = 0$ implies that both ξ_k and Δ_k are zero. Although cases in which there exists a value k_0 such that $E(k_0) = 0$ can be constructed, in general this does not happen, and ξ_k vanishes near the Fermi sphere, while Δ_k vanishes only at the infinity. So one can hope that the perturbation expansion of the ground state energy

([6]) V. V. TOLMACHEV and S. V. TIABLIKOV: *Žurn. Èksp. Teor. Fiz.*, **34**, 46 (1958).
([*]) See eq. (15), (16) and (17) of reference ([6]).
([6a]) L. VAN HOVE: *Physica*, **25**, 849 (1959).

has a finite radius of convergence in the coupling constant; this point must be carefully studied before having a rigorous theory.

The problem of finding the first non-vanishing correction to the excitation spectrum will be treated in a following paper.

2. – Formalism.

Consider the Hamiltonian:

$$H = T + V \tag{1}$$

with (δ is the Kronecker symbol):

$$T = \sum_{ks} (\varepsilon_k - \lambda) a_{ks}^+ a_{ks}, \tag{2}$$

$$V = -\tfrac{1}{2} \sum_{k_1 k_2 k_3 k_4 s_1 s_2} G_{k_1 k_4} \delta_{k_1 + k_2, k_3 + k_4} a_{k_1 s_1}^+ a_{k_2 s_2}^+ a_{k_3 s_2} a_{k_4 s_1}. \tag{3}$$

We can also write:

$$V = \tfrac{1}{4} \sum_{\substack{k_1 k_2 k_3 k_4 \\ s_1 s_2 s_3 s_4}} a_{k_1 s_1}^+ a_{k_2 s_2}^+ \langle k_1 s_1, k_2 s_2 | V | k_3 s_3, k_4 s_4 \rangle a_{k_3 s_3} a_{k_4 s_4}, \tag{4}$$

with

$$\langle k_1 s_1, k_2 s_2 | V | k_3 s_3, k_4 s_4 \rangle = \tag{5}$$

$$- \left[\frac{G_{k_1 k_3} + G_{k_2 k_4}}{2} \delta_{s_1 s_3} \delta_{s_2 s_4} - \frac{G_{k_1 k_4} + G_{k_2 k_3}}{2} \delta_{s_1 s_4} \delta_{s_2 s_3} \right] \delta_{k_1 + k_2, k_3 + k_4}.$$

We will suppose that:

$$G_{k_1 k_2} = G_{k_2 k_1}. \tag{6}$$

The canonical transformation is:

$$\begin{cases} a_{k\uparrow} = u_k \alpha_k + v_k \beta_k^+, \\[2mm] a_{-k\downarrow} = u_k \beta_k - v_k \alpha_k^+, \end{cases} \tag{7}$$

$$\begin{cases} \alpha_k = u_k a_{k\uparrow} - v_k a_{-k\downarrow}^+ \\[2mm] \beta_k = u_k a_{-k\downarrow} + v_k a_{k\uparrow}^+ \end{cases} \tag{8}$$

with u_k, v_k determined by the equations:

$$(9) \qquad u_k^2 - v_k^2 = 1 \, ,$$

$$(10) \qquad 2\xi_k u_k v_k - \varDelta_k(u_k^2 - v_k^2) = 0 \, ,$$

$$(11) \qquad \xi_k = \varepsilon_k - \lambda + \sum_{k's'} v_{k'}^2 \langle ks, \, k's' \, | \, V \, | \, k's', \, ks \rangle = \varepsilon_k - \lambda + \sum_{k'} v_{k'}^2 (G_{kk'} - G_{kk} - G_{k'k}) \, ,$$

$$(12) \qquad \varDelta_k = - \sum_{k'} \langle k\uparrow, \, -k\downarrow \, | \, V \, | - k'\downarrow, \, k'\uparrow \rangle u_{k'} v_{k'} = \sum_{k'} \tfrac{1}{2}(G_{kk'} + G_{-k,-k'}) u_{k'} v_{k'} \, .$$

Let us write:

$$(13) \qquad\qquad\qquad H = H_0 + H_{\text{int}} \, ,$$

with

$$(14) \qquad H_0 = U + \sum_k E_k(\alpha_k^+ \alpha_k + \beta_k^+ \beta_k) \, ,$$

$$(15) \qquad U = \sum_{ks} \left(\varepsilon_k - \lambda + \tfrac{1}{2} \sum_{k's'} \langle ks, \, k's' \, | \, V \, | \, k's', \, ks \rangle v_{k'}^2 \right) v_k^2 - \sum_k \varDelta_k u_k v_k =$$

$$= \sum_k \left[2(\varepsilon_k - \lambda) + \sum_{k'} v_{k'}^2 (G_{kk'} - G_{kk} - G_{k'k'}) \right] v_k^2 - \sum_k \varDelta_k u_k v_k \, ,$$

$$(16) \qquad E_k = + \sqrt{\xi_k^2 + \varDelta_k^2} \, .$$

It can be easily seen from the equations (11), (15) that the normal solution:

$$(17) \qquad\qquad\qquad \varDelta_k = u_k v_k = 0$$

is equivalent to a Hartree-Fock self-consistent approximation.

In order to give a simple form to H_{int} we introduce a new variable η_{ks} [*] defined in the following way:

$$(18) \qquad\qquad\qquad \eta_{ks} = u_k a_{ks} - v_{ks} a_{-k,-s}^+ \, ,$$

with $v_{k\uparrow} = v_k$ and $v_{k\downarrow} = -v_k$.

Eq. (18) is equivalent to the equations (7), (8), with:

$$\alpha_k = \eta_{k\uparrow} \qquad \text{and} \qquad \beta_k = \eta_{-k\downarrow} \, .$$

[*] This variable has been introduced by VALATIN (reference [4], eq. (3a)) with the symbol ξ_k^+.

From (18) one can write:

$$(19) \qquad a_{ks} = u_k \eta_{ks} + v_{ks} \eta^+_{-k,-s} ,$$

$$(20) \qquad H_0 = U + \sum_{ks} E_k \eta^+_{ks} \eta_{ks} .$$

Substitution into (1) gives:

$$(21) \quad H = \sum_{ks} (\varepsilon_k - \lambda)(u_k \eta^+_{ks} + v_{ks} \eta_{-k,-s})(u_k \eta_{ks} + v_{ks} \eta^+_{-k,-s}) -$$

$$- \frac{1}{2} \sum_{\substack{k_1 k_2 k_3 k_4 \\ s_1 s_2}} G_{k_1 k_4} \delta_{k_1 + k_2, k_3 + k_4}(u_{k_1} \eta^+_{k_1 s_1} + v_{k_1 s_1} \eta_{-k_1, -s_1}) \cdot$$

$$\cdot (u_{k_2} \eta^+_{k_2 s_2} + v_{k_2 s_2} \eta_{-k_2, -s_2})(u_{k_3} \eta_{k_3 s_2} + v_{k_3 s_2} \eta^+_{-k_3, -s_2})(u_{k_4} \eta_{k_4 s_1} + v_{k_4 s_1} \eta^+_{-k_4 -s_1}) .$$

It is well known (see *f.i.* ref. [5]) that H_{int} is a sum of normal products of four operators. Furthermore we will show that H_{int} can be written simply as follows:

$$(22) \quad H_{\text{int}} = -\frac{1}{2} \sum_{\substack{k_1 k_2 k_3 k_4 \\ s_2 s_2}} G_{k_1 k_4} \delta_{k_1 + k_2, k_3 + k_4} \, \mathrm{N_{form}} \Big\{ (u_{k_1} \eta^+_{k_1 s_1} + v_{k_1 s_1} \eta_{-k_1, -s_1})(u_{k_2} \eta^+_{k_2 s_2} + v_{k_2 s_2} \eta_{-k_2, -s_2}) \cdot$$

$$\cdot (u_{k_3} \eta_{k_3 s_2} + v_{k_3 s_2} \eta^+_{-k_3, -s_2})(u_{k_4} \eta_{k_4 s_1} + v_{k_4 s_1} \eta^+_{-k_4, -s_1}) \Big\} ,$$

where $\mathrm{N_{form}}$ means « normal form ». Therefore the product must be written with all the creation operators standing to the left of the destruction operators.

Let us call $|\Phi_0\rangle$ the vacuum of the quasi particles; we have:

$$(23) \qquad \eta_{ks} |\Phi_0\rangle = 0 .$$

Defining the contraction $\langle AB \rangle$ of two operators A, B as $\langle \Phi_0 | AB | \Phi_0 \rangle$, we can write:

$$(24) \quad \begin{cases} \langle \eta_{k_1 s_1} \eta_{k_2 s_2} \rangle = \langle \eta^+_{k_1 s_1} \eta^+_{k_2 s_2} \rangle = \langle \eta^+_{k_1 s_1} \eta_{k_2 s_2} \rangle = 0 , \\[2mm] \langle \eta_{k_1 s_1} \eta^+_{k_2 s_2} \rangle = \delta_{k_1 k_2} \delta_{s_1 s_2} . \end{cases}$$

In order to prove the formula (22) we apply to the Hamiltonian H the Wick-Dyson theorem (ref [7]) which allows the decomposition of a product of operators into a sum of normal products. Obviously for two operators we have:

$$(25) \qquad AB = \langle AB \rangle + \mathrm{N_{form}}(AB) ,$$

[7] F. J. DYSON: *Phys. Rev.*, **82**, 428 (1951).

and for four fermion operators A, B, C, D:

(26) $$ABCD = N_4 + N_2 + N_0 \, ,$$

where

$$N_4 = N_{\text{form}}(ABCD) \, ,$$
$$N_2 = N_{\text{form}}\{AB\langle CD\rangle - AC\langle BD\rangle + AD\langle BC\rangle + $$
$$+ \langle AB\rangle CD - \langle AC\rangle BD + \langle AD\rangle BC\} \, ,$$
$$N_0 = \langle AB\rangle\langle CD\rangle - \langle AC\rangle\langle BD\rangle + \langle AD\rangle\langle BC \, .$$

Applying the decomposition (25), (26) to the R.H.S. of eq. (21), it is easy to verify that:

1) Terms like N_0 contribute only to U.

2) Terms like N_2 contribute either to $\sum_{l,s} E_k \eta_{ks}^+ \eta_{ks}$, or to H_{20} (eq. (93) of BELIAEV). And because H_{20} vanishes in virtue of eq. (10), we are left with terms N_4. Eq. (22) is then proved.

It is possible now to make a perturbation expansion of the energy shift ΔE_0 of the ground state $|\Phi_0\rangle$ using the Goldstone formula (ref. ([8,9])):

(27) $$\Delta E_0 = - \langle \Phi_0 | [-H_{\text{int}} + H_{\text{int}}(H_0 - U)^{-1} H_{\text{int}} \ldots]_C | \Phi_0\rangle \, ,$$

where the symbol C means that only connected graphs contribute.

As for the graph technique we must notice that in this theory:

a) There are no lines with the arrow pointing to the right (in the terminology of Hugenholtz), the ground states $|\Phi_0\rangle$ being the vacuum of the quasi particles.

b) There are no lines with the two end points at the same vertex; in fact the interaction H_{int} is written in normal form, and from the formulae (24) we see that all the contractions between two operators of the same H_{int} vanish.

The first non-vanishing term of the R.H.S. of eq. (27) is:

(28) $$\Delta E_0^{(2)} = - \langle \Phi_0 | H_{\text{int}}(H_0 - U)^{-1} H_{\text{int}} | \Phi_0\rangle \, .$$

We have omitted the symbol C, because at the second order all the graphs must be connected, in virtue of the observation b).

([8]) J. GOLDSTONE: Proc. Roy. Soc., A **239**, 627 (1957).
([9]) N. M. HUGENHOLTZ: Physica, **23**, 481 (1957).

We get:

$$(29) \quad \Delta E_0^{(2)} = -\frac{1}{4} \sum_{\substack{k_1 k_1' q_1 \\ s_1 s_1'}} \sum_{\substack{k k' q \\ s s'}} G_{k_1 k_1'} G_{k k'} u_k u_{q-k} v_{q-k',s'} v_{k's} u_{k_1} u_{q_1-k_1'} v_{q_1-k_1,s_1} v_{k_1 s_1} \cdot$$

$$\cdot \frac{\langle \varphi_0 | \eta_{-k_1,-s_1} \eta_{k_1-q_1,-s_1'} \eta_{q_1-k_1',s_1'} \eta_{k's_1}^+ \eta_{q-k,s'}^+ \eta_{k'-q,-s'}^+ \eta_{-k',-s}^+ | \varphi_0 \rangle}{E_k + E_{q-k} + E_{q-k'} + E_{k'}} .$$

In order to compute the vacuum expectation value appearing in eq. (29) we can use again the Wick-Dyson theorem. But it is simpler to use the methods of Caianiello ((ref. [10])) which allow us to write the vacuum expectation value of a product of m creation operators $\eta_{k_1}^- \eta_{k_2}^- \cdots \eta_{k_i}^+$ and m annihilation operators $\eta_{h_1} \eta_{h_2} \cdots \eta_{h_m}$ in the following way:

$$(30) \quad \langle \Phi_0 | \eta_{h_1} \eta_{h_2} \cdots \eta_{h_m} \eta_{k_1}^+ \eta_{k_2}^+ \cdots \eta_{k_m}^+ | \Phi_0 \rangle = (-1)^{\binom{m}{2}} \operatorname{Det}(a_{rs}) ,$$

where $\operatorname{Det}(a_{rs})$ is the determinant $m \times m$ whose elements a_{rs} are the contractions:

$$(31) \quad a_{rs} = \langle \Phi_0 | \eta_{h_r} \eta_{k_s}^+ | \Phi_0 \rangle .$$

Therefore, putting:

$$(32) \quad h_k = v_k^2 ; \qquad j_k = 1 - h_k = u_k^2 ; \qquad \chi_k = u_k v_k$$

and using the formulae (24), (30), we can write the expression (29) as:

$$(33) \quad \Delta E_0^{(2)} = \frac{1}{2} \sum_{k k' q} \frac{G_{k k'} \left(\sum_{i=1}^{6} G_i \right)}{E_k + E_{q-k} + E_{q-k'} + E_{k'}} ,$$

where

$$(34) \quad \begin{cases}
G_1 = (-2 G_{k k'} + G_{k,q-k'} + G_{q-k,k'} - 2 G_{q-k,q-k'}) \chi_k \chi_{q-k} \chi_{q-k'} \chi_{k'} , \\
G_2 = (G_{k,q-k'} + G_{-k,q-k} + G_{-k',q-k'} + G_{k',q-k}) \chi_k j_{q-k} \chi_{q-k} h_{k'} , \\
G_3 = (G_{-k,q-k} - 2 G_{k k'} + G_{q-k',-k'} - 2 G_{q-k',q-k}) \chi_k j_{q-k} h_{q-k'} \chi_{k'} , \\
G_4 = (G_{q-k,-k'} + G_{q-k,k'} + G_{q-k',k} + G_{q-k',-k'}) j_k \chi_{q-k} h_{q-k} \chi_{k'} , \\
G_5 = (-2 G_{q-k',q-k} + G_{q-k',k} + G_{k',q-k} - 2 G_{k'k}) j_k j_{q-k} h_{q-k'} h_{k'} , \\
G_6 = (-2 G_{q-k,q-k'} + G_{q-k,-k} - 2 G_{k'k} + G_{-k',q-k'}) j_k \chi_{q-k} \chi_{q-k'} h_{k'} .
\end{cases}$$

[10] E. R. CAIANIELLO: *Nuovo Cimento*, **10**, 1634 (1953).

If $G_{k'k} = G_{|k-k'|}$, we get the simplified expressions:

(35)
$$\begin{cases} G_1 = (2\,G_{|q-k-k'|} - 4\,G_{|k-k'|})\,\chi_k\,\chi_{q-k}\,\chi_{q-k'}\,\chi_{k'}\,, \\[2mm] G_2 = (2\,G_{|q-k-k'|} + 2\,G_{|q|})\quad \chi_k\,j_{q-k}\,\chi_{q-k'}\,h_{k'}\,, \\[2mm] \text{etc.} \end{cases}$$

3. – Application to the B.C.S. theory.

We shall make the same assumption as BELIAEV, *i.e.*:

(36)
$$\begin{cases} \langle k\uparrow, -k\downarrow\,|V\,|-k'\downarrow\,k'\uparrow\rangle = -\,G \qquad \text{for} \qquad |\xi_k|,\ |\xi_{k'}| < \omega \quad (*), \\[2mm] \qquad\qquad\qquad\qquad = 0 \qquad \text{otherwise}\,. \end{cases}$$

We make the following hypothesis ($\varrho\,d\varepsilon$ is the number of states in an interval $d\varepsilon$ near the Fermi surface, and ε_{k_F} is the Fermi energy):

(37)
$$G\varrho < 1\,,$$

(38)
$$\omega \ll \varepsilon_{k_F}\,.$$

As well known we have:

(39)
$$\Lambda_k = \Lambda = \omega\left(\sinh\frac{1}{G\varrho}\right)^{-1}, \qquad\qquad \text{for } |\xi_k| < \omega\,,$$

Calling A, R, B the zones of the k space so that $\xi_k < -\omega$, $-\omega < \xi_k < +\omega$ and $\xi_k > +\omega$ respectively, we get the following table:

A	R	B
$\Lambda_k = 0$	$\Lambda_k = \Lambda$	$\Lambda_k = 0$
$E_k = |\xi_k|$	$E_k = +\sqrt{\xi_k^2 + \Lambda^2}$	$E_k = \xi_k$
$h_k = 1$	$h_k = \dfrac{1}{2}\left(1 - \dfrac{\xi_k}{E_k}\right)$	$h_k = 0$
$j_k = 0$	$j_k = \dfrac{1}{2}\left(1 + \dfrac{\xi_k}{E_k}\right)$	$j_k = 1$
$\chi_k = 0$	$\chi_k = \dfrac{1}{2}\dfrac{\Lambda}{E_k}$	$\chi_k = 0$

(*) We must assume that ξ_k is a single-valued function of ε_k.

We have, for all k:

(40)
$$E_k \geqslant \varDelta .$$

From the assumptions (37), (38), we see that $\varDelta \ll \varepsilon_{k_F}$. Therefore we can neglect all the G's containing a factor χ. So we are left with G_5 and correspondingly we have:

(41)
$$\varDelta E_0^{(2)} \simeq \frac{1}{2} \sum_{kk'q} \frac{G_{kk'} \, G_5}{E_k + E_{q-k} + E_{q-k'} + E_{k'}} .$$

It is easy to see that $\varDelta E_0^{(2)}$ can be written as follows:

(42) $\varDelta E_0^{(2)} = -\dfrac{1}{4} \sum_{\substack{k_1 k_2 k_3 k_4 \\ s_1 s_2 s_3 s_4}} \dfrac{\langle k_1 s_1, k_2 s_2 \, | \, V \, | \, k_3 s_3, k_4 s_4 \rangle \langle k_4 s_4, k_3 s_3 \, | \, V \, | \, k_2 s_2, k_1 s_1 \rangle}{E_{k_1} + E_{k_2} + E_{k_3} + E_{k_4}} \, j_{k_1} j_{k_2} h_{k_3} h_{k_4} .$

We now make the approximation of substituting $|\xi_k|$ to $E_k = +\sqrt{\xi_k^2 + \varDelta^2}$ in the denominators and in the definitions of h_k and J_k. We are allowed to make this approximation if the contribution of small denominators can be neglected. For instance if E_k is a function of $|k|$ only, the contribution of the zones of the q and k spaces giving rise to small denominators is very small and the approximation is allowed. So h_k and J_k become simply the step-functions:

(43)
$$\begin{cases} h_k = \dfrac{1}{2}\left(1 - \dfrac{\xi_k}{|\xi_k|}\right), \\[2mm] J_k = \dfrac{1}{2}\left(1 + \dfrac{\xi_k}{|\xi_k|}\right). \end{cases}$$

From eq. (42) we see that the value of $\varDelta E_0^{(2)}$ is equal, in this approximation, to the contribution of the graph:
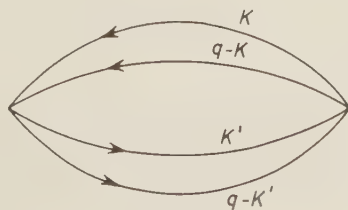
Graph F.

calculated with a Hartree-Fock self-consistent method (see eq. (17)).

This result could have been expected, because if the interaction between pairs of zero total momentum is effective only on a small shell, the normal state differs very little from the se called « superconducting state ».

The energy difference between the normal state and the superconducting state is given by (BELIAEV, eq. (127)):

$$(44) \qquad\qquad E_N - E_s = \varrho\omega^2(\text{ctg } h\eta - 1) .$$

The second order correction to this energy can be calculated from our formula (33), subtracting the contribution of the graph F.

We notice that taking the term $\sum_{ks} (\varepsilon_k - \lambda)a_{ks}^+ a_{ks}$ as the unperturbed Hamiltonian, there is another diagram to be considered, i.e.:
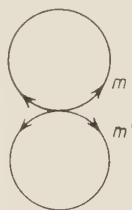
As we have seen, this diagram cannot appear in our approach.

For the same reason we do not have, at the third order, the diagram:
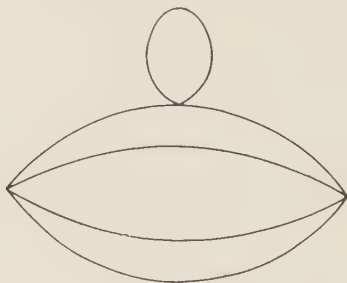
The canonical transformation cancels, both in the cases $\varDelta \neq 0$ and $\varDelta = 0$, all the diagrams containing lines with the two end points at the same vertex (*).

Something analogous happens in quantum electrodynamics, where these diagrams are eliminated by Heisenberg's prescription.

(*) In the case $\varDelta = 0$ we get again the well-known property of the self-consistent method.

* * *

RIASSUNTO

Viene eseguita una trasformazione canonica, identica a quella di Bogoljubov e Valatin; viene eseguito inoltre uno sviluppo perturbativo dell'energia dello stato fondamentale, prendendo come Hamiltoniana imperturbata il termine che descrive « quasi particelle » libere. La perturbazione viene scritta in forma normale: ne segue che la trasformazione canonica cancella, a tutti gli ordini, diagrammi contenenti linee che escono ed entrano dallo stesso vertice. È svolta inoltre una semplice discussione sul valore dell'energia dello stato fondamentale nel caso considerato dalla teoria di Bardeen, Cooper e Schrieffer.

γ-Ray Spectroscopy of Artificial Radioactive Samples from Atmospheric Air.

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1. – Introduction.

Recently the study of the atmospheric radioactivity has notoriously acquired new reasons of interest and points of research. In fact the fission products dispersed in the atmosphere from the nuclear and thermonuclear explosions (and from nuclear reactors but in almost negligible amounts, because of the precautionary measures in order to avoid radionuclide losses) have added themselves to the natural radionuclides ($^{222}_{86}$Em and $^{220}_{86}$Em diffusing from the ground, and their decay products) and to traces of some radionuclides originated by the cosmic radiation through the atmosphere (first of all, for its interesting applications, the $^{14}_{6}$C; furthermore, among others $^{3}_{1}$H, $^{7}_{4}$Be, ...).

The presence of these artificial radionuclides in the atmosphere, raises on one hand problems related to the atmospheric pollution, to the maximum permissible levels of exposure to radiations and to radiation protection, on the other hand paves the way for new researches, using the radioactive tracers method, on a whole of atmospheric phenomena (as, for instance, the exchange processes among the various layers of the atmosphere and between the two hemispheres, atmospheric circulation, ...).

The natural radionuclides are coming, as said before, from the ground through the $^{222}_{86}$Em and $^{220}_{86}$Em and because of the relatively short half-life of the first and very short of the second (and of the scavenging of the atmosphere from the solid radioactive pollutions) their concentration decreases quickly with the height, so that the presence of these radionuclides concerns essentially the lower layers of the atmosphere and feels the repercussion of the atmospheric phenomena which take place there.

On the contrary the products of a nuclear or thermonuclear explosion, if the energy is less than 1 megaton (T.N.T.), concern essentially the troposphere while if the energy is greater than 1 megaton they penetrate into

the lower stratosphere to a depth according to the conditions of the explosion and the energy released [1].

The time required in order that the smaller radioactive dust particles (diameter $< 1 \cdot 10^{-6}$ m), originated in the explosion and which stay easily in suspension in the air, come back to the ground is of order of some months in the first case and of several years in the second one, because of the slowness of the exchange processes between stratosphere and troposphere [2].

Therefore the sufficiently long lived nuclides (f.i. $^{90}_{38}Sr$, $T = 28$ y; $^{137}_{55}Cs$, $T = 30$ y) play an important part as tracers in the upper atmospheric phenomena and in the relation between the upper and lower atmosphere.

Also from this point of view, it becomes very interesting, in order to integrate, if possible, the measurements of the total artificial atmospheric radio-

of γ-ray spectroscopy, are developing and already give encouraging results.

Researches on this subject are the aim of the present paper.

2. - Collection of the radioactive samples.

The eight examined samples were collected in Naples between October 1958 and June 1959, at the Gabinetto di Meteorologia ed Oceanografia of the Istituto Universitario Navale, by means of filtration of the air at the ground, some on Millipore AA paper (samples $1 \div 6$) some on Watman 40 paper (samples 7 and 8). The preparation of samples was performed by Prof. ALIVERTI and Dr. DE MAIO in connection with the AGI program of researches.

The essential data concerning the collection of the samples are given in Table I.

TABLE I.

Sample	Date of the collection	Duration of the collection (h)	m³ of filtered air at °C and 760.0 mm Hg
1	21st October 1958	9	16.7
2	2nd December 1958	10	14.9
3	16th and 17th February 1959	20	26.5
4	9th and 10th March 1959	24	90.8
5	4th and 5th May 1959	25.4	81.2
6	12th and 13th May 1959	24	73.4
7	19th and 20th June 1959	30	342
8	22nd and 23rd June 1959	31.5	439

activity, to single out the various radionuclides, almost the prominent ones.

To this purpose, together with the processes of radiochemical analysis, the processes founded on the analysis of the radiations, essentially by means

3. - Analysis of the samples.

The γ radiation emitted by the sample was analyzed by means of an apparatus consisting in: NaI(Tl) crystal ($1\frac{1}{2}$ in. $\times 1\frac{1}{2}$ in.) followed by a 53 AVP photomultiplier; RIDL 100 channel analyzer.

The apparatus remained unchanged in all the components for all the duration of the measurements.

[1] W. F. LIBBY: Proc. Nat. Acad. Sci., 42, 945 (1956); 43, 758 (1957).

[2] C. JEHANNO and J. LABEYRIE: Journ. de Phys. et Rad., 20, 702 (1959).

The apparatus's calibration was periodically verified using standard sources; the photoelectric peaks of the γ-rays emitted by these sources were always on the same channels.

All the samples were analyzed in the range of energies between 0.25 and 1.70 MeV (15.5 keV/channel); moreover the sample n. 8 was analyzed also in the range of low energies between 0.03 and 0.70 MeV (7.15 keV/channel).

4. – Results.

4˙1. *Search for γ-rays and measurements of their energy.* – In the range of

energies (0.25 ÷ 1.70) MeV we have observed the prominent peaks given in Table II.

Moreover, we can observe just to the right of the 0.75 MeV peak another somewhat less prominent peak, due to the $^{140}_{57}$La (0.815 MeV).

In the range of low energies we can observe the three further peaks given in Table III.

Obviously a peak can be resultant of some γ-rays originated by different components; in this case the relative intensities of the various components change (and for some can also cancel) according to the time elapsed between

TABLE II.

Energy (MeV)	attributed to
0.36	$^{140}_{56}$Ba (0.310 MeV; $T=12.8$ d); $^{140}_{57}$La (0.328 MeV; $T=40.2$ h); $^{131}_{53}$I (0.364 MeV; $T=8.05$ d)
0.49	$^{140}_{57}$La (0.487 MeV; $T=40.2$ h); $^{103}_{44}$Ru (0.498 MeV; $T=41.0$ d); $^{140}_{56}$Ba (0.54 MeV; $T=12.8$ d)
0.66	$^{137}_{55}$Cs (0.661 MeV; $T=30$ y)
0.75	$^{95}_{40}$Zr (0.722 MeV; 0.754 MeV; $T=63.3$ d); $^{95}_{41}$Nb (0.768 MeV; $T=35$ d)
1.60	$^{140}_{57}$La (1.596 MeV; $T=40.2$ h)

TABLE III.

Energy (MeV)	attributed to
0.03 ÷ 0.04	$^{140}_{56}$Ba (0.030 MeV; $T=12.8$ d); $^{144}_{58}$Ce (0.0337 MeV; 0.054 MeV; $T=290$ d); $^{103}_{45}$Rh^{m} (0.040 MeV; $T=54$ d)
0.08 ÷ 0.09	$^{144}_{58}$Ce (0.0807 MeV; 0.100 MeV; $T=290$ d); $^{133}_{54}$Xe (0.081 MeV; $T=5.27$ d); $^{147}_{60}$Nd (0.092 MeV; $T=11.3$ d); $^{140}_{57}$La (0.093 MeV; $T=40.2$ h); $^{129}_{52}$Te^{m} (0.106 MeV; $T=33$ d)
0.15	$^{141}_{58}$Ce (0.145 MeV; $T=32$ d); $^{140}_{57}$Ba (0.160 MeV; $T=12.8$ d)

the nuclear event which originated the radioactive material and the analysis of the sample, in relation to the half-life of the same nuclides and of their eventual parents.

4˙2. *Samples* n. 1, 2, 3, 5, 6. — For each of these samples we have realized a single analysis of the γ spectrum, whether for their weak activity which does not allow reliable measurements after rather long time intervals from the date of the collection, or, as concerns the first three, because their collection and examination coincided with the preliminary orientative stage of these researches.

The 0.36, 0.49, 0.66, 0.75 MeV peaks are present in the γ spectra of the samples 5 and 6; the 0.75 MeV peak also in the γ spectrum of the sample n. 3.

The other three samples (n. 4, 7, 8) were analyzed in several sets of measurements at regular time intervals.

The more careful analysis of the time dependence of the activity has been effected on the sample n. 8; therefore we report first of all the experimental results on this sample. It was obtained by filtration of a greater quantity of air and had the higher total activity, however the sample n. 4 had the higher activity per m³ of filtered air.

4˙3. *Sample* n. 8. — Sample n. 8 (realized the 22nd and 23rd June 1959, see Table I) was analyzed in the range of energies between 0.25 and 1.70 MeV in three sets of measurements carried out during the time included between 26th June and 2nd July, 16th and 24th July, 24th and 27th August, that is, assuming the centres of the time intervals, after about 6, 27 and 63 days from the collection.

This sample was also analyzed in the range of energies between 0.03 and 0.70 MeV in two sets of measurements during the time included between 28th July and 4th August, 27August and 2nd September, that is after about

Fig. 1 *a*).

38 and 67 days from the collection.

Each set of measurements corresponds to ten hours of effective measurements (total minus accidental).

are indicated through vertical arrows and are in MeV.

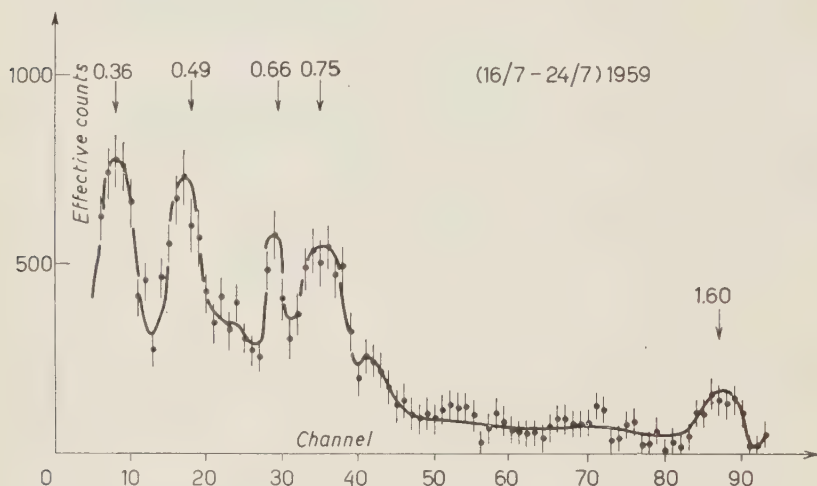We can remark at once that there are considerable differences in the time

Fig. 1 b).

Fig. 1 a), b) and c) show the experimental results for sample n. 8. We may observe the total effective number of

dependence of the size of the various peaks.
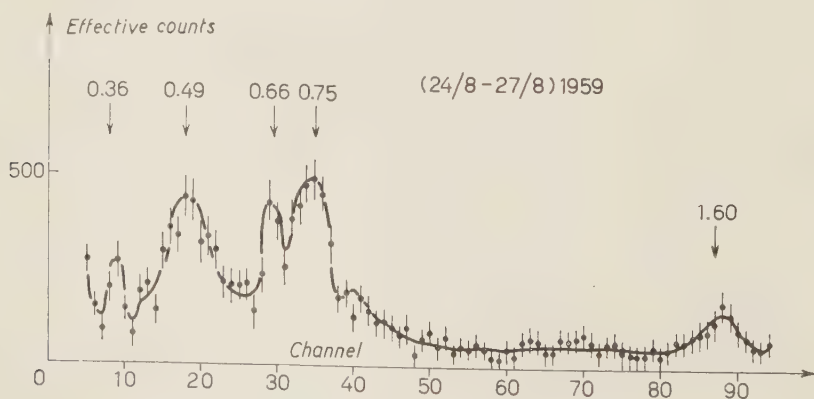
We can deduce therefore a quick

Fig. 1 c).

pulses as a function of the channel number for the three sets of measurements in the range of energies between 0.25 and 1.70 MeV. In these spectra, as well as in the following ones, energies

variation of the relative intensities of the corresponding lines; we may then suppose that the sample consists of radioactive material prevalently of recent date in rather quick evolution. Obviously

the γ-ray peaks of the radionuclides having a shorter half-life, present the most remarkable variations.

This is the case of the 0.36 MeV peak, whose height decreases very quickly with time; this peak is due to the following nuclides: $^{131}_{53}$I ($T = 8.05$ d), $^{140}_{56}$Ba ($T = 12.8$ d) and its daughter product $^{140}_{57}$La ($T = 40.2$ h).

The 1.60 MeV peak, due to $^{140}_{57}$La, decreases hardly more slowly than the 0.36 MeV peak.

We can observe a more slow variation of the 0.49 MeV peak due to $^{140}_{56}$Ba, $^{140}_{57}$La and $^{103}_{44}$Ru ($T = 41.0$ d). On the contrary the 0.75 MeV peak due to $^{95}_{40}$Zr ($T = 63.3$ d) and $^{95}_{41}$Nb ($T = 35$ d) belonging to the same decay chain, and the 0.66 MeV due to $^{137}_{55}$Cs ($T = 30$ y) are particularly persistent. The small variation of the last is only apparent (during the time of our sets of measurements) and is rather to be ascribed

In order to make a more accurate quantitative examination of the spectra we have tried to estimate the Compton background contribution of the various monoenergetic γ rays and we have subtracted them from the experimental distribution.

We have taken into consideration the photoelectric and Compton contributions to the absorption processes respectively as a function of the γ rays energy in a cylindrical NaI(Tl) crystal having the same size of the crystal used in our researches and the energy distributions of Compton electrons observed with the same crystal using monoenergetic γ rays emitted by standard sources ([3]).

Fig. 2 shows, for instance, the curve obtained subtracting the Compton background from the experimental results of Fig. 1 a) according to what is said above.
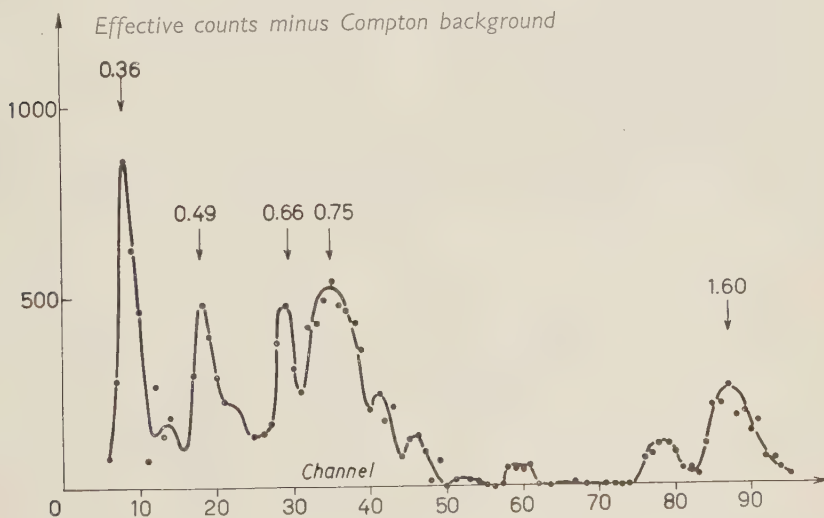

Fig. 2.

to the variations of the background originated essentially by the Compton distribution which can contribute to the apparent size of the peak (and eventually to the statistical fluctuations).

([3]) P. R. BELL: Beta and Gamma-Ray Spectroscopy, edited by K. SIEGBAHN (Amsterdam, 1955); D. MAEDER and V. WINTERSTEIGER: Phys. Rev., 87, 537 (1952); Helv. Phys. Acta, 25, 465 (1952); 27, 3 (1954).

The 0.75 MeV peak, due to $^{95}_{40}$Zr and $^{95}_{41}$Nb, is assumed as a reference peak. First of all we have compared the time dependence of the intensity of this peak with the theoretical curve of total activity of the two said nuclides as a whole.

Afterwards we have calculated the intensities of the other peaks with reference to the said peak and we have compared their time dependences with those obtained in the same manner from the Table reported by SCAFATI ([4]). Obviously for the composite peaks we have added up the contributions due to the various components.

Fig. 3 shows the γ spectrum in the range of energies between 0.03 and 0.70 MeV corresponding to the first set of these measurements.

The experimental behaviours are in satisfactory agreement with those to be expected if we suppose about forty days have elapsed between the event (or the whole of near events) which originated the radioactive material and the date of the first analysis.

This evaluation is confirmed by the decay of the total activity.

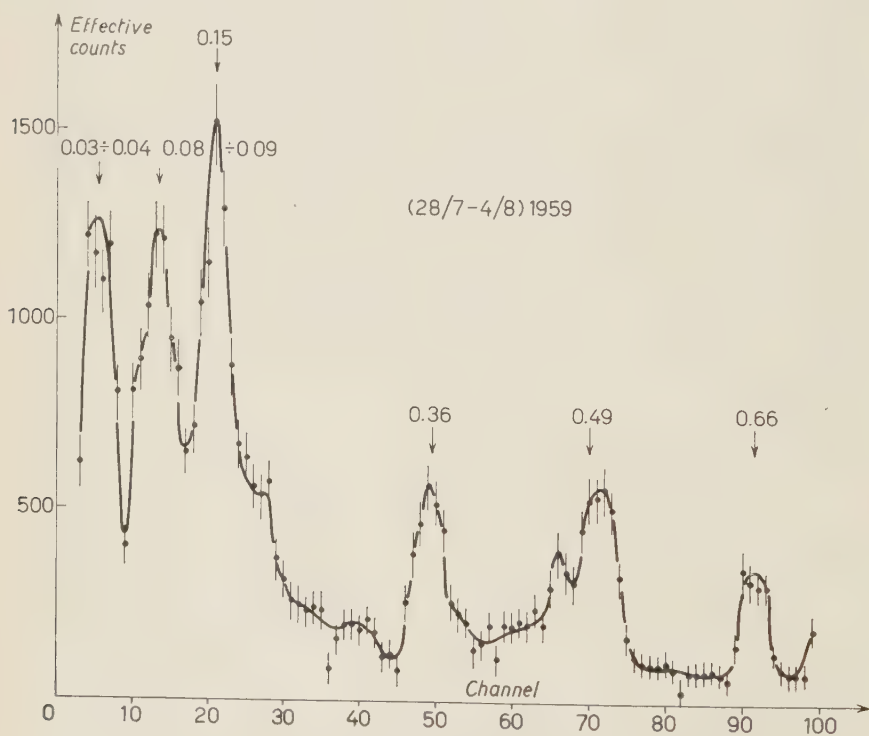Precisely if we deduce the total activity simply by adding up all the

Fig. 3.

Moreover we have extended the same quantitative study to the γ spectra obtained in the low energies range.

([4]) A. SCAFATI: *Minerva Nucleare*, 3, 16 (1959).

pulses in each channel of the analyzer, and we report the inverse of the total activity against the time, we obtain a straight line with a good approximation; the extrapolation of the straight line meet the time axis at the date of the event corresponding to about thirty

five days before the first set of measurements.

On the other hand the NaI(Tl) crystal efficiency is different for various γ ray energies. This behaviour of the efficiency can increase the contribution to the total time dependence of the nuclides whose photoelectric peaks are in the higher efficiency range (in this case in the low energies range).

Therefore we have corrected for the efficiency the experimental distribution; the new diagram is again a straight line and the date of the event is anticipated by about only 6 days, that is about forty days before the first set of measurements.

However we must note that the presence of old radioactive material, pointed out by the peak of $^{137}_{55}$Cs, can obviously affect the date of the event which originated the bulk of the sample material increasing its apparent age and, at the same time, the uncertainty of the result.

4.4. *Sample n. 7.* – Sample n. 7 (collected on the days 19th and 20th June, see Table I) was analyzed in the range of energies between 0.25 and 1.70 MeV in two sets of measurements performed during the time included between 23rd June and 1st July, 3rd and 9th September, that is, assuming the centres of the time intervals, after about 7 and 78 days from the collection.

Fig. 4 shows the experimental results obtained in the first set of measurements in the range of energies between 0.25 and 1.70 MeV.

By comparing these experimental results with the results on Fig. 1 *a*) we can observe that the peaks are the same, but their relative intensities seem to us entirely different and indicate, in sample n. 7, a radioactive material of date somewhat less recent in respect of sample n. 8.

We have extended the quantitative examination of the experimental results of sample n. 7 following the same method used for sample n. 8.

The experimental results are in agreement with the theoretical ones if we suppose that the time interval between the event which originated the radioactive material and the respective dates of the first and the second analysis is of the order of about sixty days, and four months and a half.

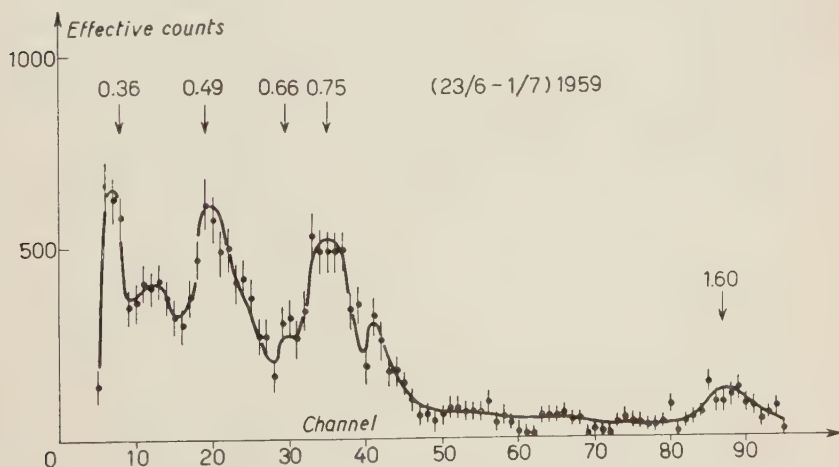This evaluation is confirmed by the total activity decay.

Fig. 4.

4.5. *Sample n. 4.* – We have already reported the results on sample n. 4 ([5]).

This sample, collected during the 9th and 10th March (see Table I) was analyzed in the range of energies between 0.25 and 1.70 MeV in three sets of measurements carried out during the time between 11th and 21st March, 8th and 16th April, 17th and 24th April.

The prominent peaks, in Table II, are present in the γ spectrum of this sample. The quantitative analysis of the time dependence of their intensities confirms the already said conclusions ([5]).

5. – Conclusions.

The accurate examination of the samples 7 and 8, said above, allowed us to deduce certain interesting conclusions about the presence of some radionuclides and the time dependence of their activity.

Particularly we point out the following fact. The relative intensities of the peaks in the spectra of the two samples 7 and 8 are very different, taking into account that the two samples have been collected at an interval of only about two days.

We are led to believe that they are carried by air masses having different histories and characteristics.

On this subject, we can notice from

([5]) G. ALIVERTI, F. DEMICHELIS and G. LO-VERA: *Nuovo Cimento*, **13**, 453 (1959).

the Italian Air Force Bulletins that on the 19th and 20th June the jet-stream was passing over Southern Italy (at the velocity of about 150 km/hour), while on the 22nd and 23rd June it was shifted further south beyond Naples and precisely above Greece and the adjoining seas.

So that in the intermediate days the edge of the jet-stream can have interested Naples. The radioactive material given up by the jet-stream, mainly at its edge, to the winds of the lower atmosphere, can therefore have contributed particularly to the collection of sample n. 8.

From the whole of the obtained results we can deduce the following conclusions. An accurate study of the artificial radionuclides present in the atmospheric air by means of γ spectroscopy, can give valuable information whether on the atmospheric pollution or as contribution to the problems concerning the exchanges between the stratosphere and the troposphere and the general atmospheric circulation.

* * *

On the Polarization of Photons Elastically Scattered by Mercury Atoms.

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We report here some theoretical evaluations on the polarization of the photons elastically scattered by Hg atoms. We take in to account both the Rayleigh and Thomson scattering (and their interference).

We have constructed — referring to a formalism introduced by Fano [1] for electrodynamic processes and which has already been applied on several occasions during the last few years — the T matrix, characteristic for the interaction and which, when applied to the state of polarization of the incident beam, will permit to obtain that of the scattered one. The polarization states have been described by means of the Stokes parameters [2] P_j ($j=0, 1, 2, 3$). P_0 represents the beam intensity, P_1/P_0 the degree of circular polarization ($P_1 > 0$ means anticlock-wise polarization), P_2/P_0 represents the degree of linear polarization referred to the scattering plane ($P_2 > 0$ means electric vector on the scattering plane), and P_3/P_0 the analogous to the previous one with reference to a plane rotated anticlock-wise by $\pi/4$.

Therefore, if $P_j^{(\text{in})}$ are the Stokes parameters of the incident beam and $P_j^{(\text{sc})}$ are those of the scattered one, we have:

$$P_\mu^{(\text{scatt})} = T_{\mu\nu} P_\nu^{(\text{inc})} .$$

Indicating with A_R and A_T, respectively for the Rayleigh and Thomson scatterings, the amplitudes for the transitions between the states of circular polarization without change in polarization and with B_R and B_T the analogous amplitudes for the transitions with change in the polarization, we have, after putting $A_R + A_T = A$ and $B_R + B_T = B$:

$$T = r_0^2 \begin{vmatrix} |A|^2 + |B|^2 & 0 & 2\,\mathrm{Re}\,AB^* & 0 \\ 0 & |A|^2 - |B|^2 & 0 & 2\,\mathrm{Im}\,AB^* \\ 2\,\mathrm{Re}\,AB^* & 0 & |A|^2 + |B|^2 & 0 \\ 0 & -2\,\mathrm{Im}\,AB^* & 0 & |A|^2 - |B|^2 \end{vmatrix} .$$

[1] U. Fano: *Journ. Opt. Soc. Am.*, **39**, 859 (1949).
[2] See, for example, W. McMaster: *Am. Journ. Phys.*, **22**, 170 (1954).

The quantities A_T and B_T are obtained, in a quite obvious way from the well known amplitudes of the Thomson scattering:

$$A_T = (Z^2/1\,840A)[(1 + \cos\theta)/2]\,,$$
$$B_T = - (Z^2/1\,840A)[(1 - \cos\theta)/2]\,.$$

For the quantities A_R and B_R we have utilized the numerical values of the amplitudes of the Rayleigh scattering from K electrons of Hg atoms which in a series of articles by S. BRENNER, G. E. BROWN, D. F. MAYER and J. B. WOODWARD [3,5]

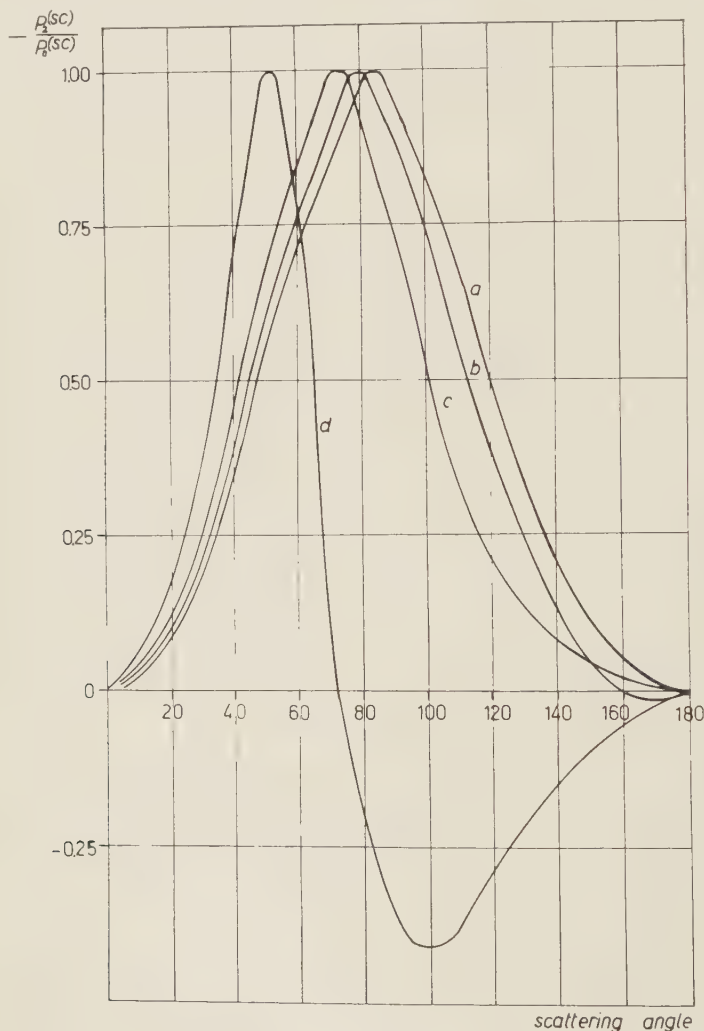Fig. 1. – Degree of linear polarization of the scattered beam for incident beam unpolarized.
a) $E_\gamma = 0.32$;   b) $E_\gamma = 0.64$;   c) $E_\gamma = 1.28$;   d) $E_\gamma = 2.56$ mc².

[3] S. BRENNER, G. E. BROWN and J. B. WOODWARD: Proc. Roy. Soc., A **227**, 59 (1955).
[4] G. E. BROWN and D. F. MAYERS: Proc. Roy. Soc., A **234**, 397 (1956).
[5] G. E. BROWN and D. F. MAYERS: Proc. Roy. Soc., A **242**. 89 (1957).

are given as functions of the scattering angle and for the values of the energy of 0.32, 0.64, 1.28, $2.56\,mc^2$. The contribution of L electrons has not been taken into account.

By knowing the matrix T one can deduce all the information on the effects of polarization in the process under consideration. In Fig. 1 are shown, with reference only to the Rayleigh scattering by K electrons of Hg atoms, the plots for the degree of linear polarization $- P_2^{(sc)}/P_0^{(sc)}$ of the scattered beam, in the case of the incident beam unpolarized, and in Fig. 2 the plots for the degree of circular polarization $- P_1^{(sc)}/P_0^{(sc)}$ in the case of the incident beam linearly polarized to $\pi/4$ with respect to the scattered plane.
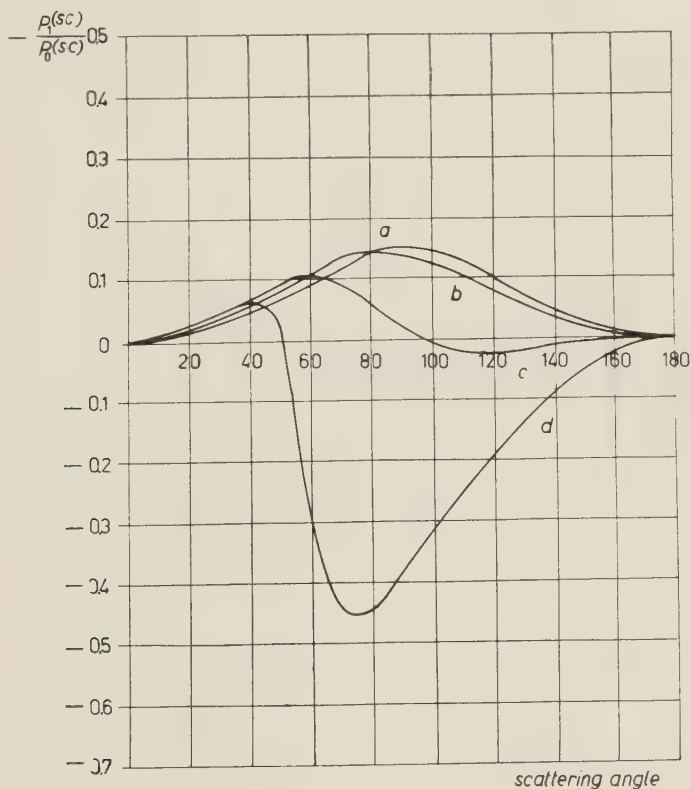
Fig. 2. – Degree of circular polarization of the scattered beam for incident beam linearly polarized at $45°$ with the scattering plane. a) $E_\gamma = 0.32$; b) $E_\gamma = 0.64$; c) $E_\gamma = 1.28$; d) $E_\gamma = 2.56\,mc^2$.

Fig. 3 indicates the degree of linear polarization at $2.56\,mc^2$ taking into account the coherent super-imposition of the Rayleigh scattering on Thomson scattering. One can see how the Thomson scattering notably modifies the degree of polarization eliminating its change of sign for $\theta = 75°$.

In a recent measurement of the degree of linear polarization effected by B. S. Sood [6] for $\theta = 90°$ and the energy of 1.25 MeV a percentage of polarization of $- (6 \pm 2.5)\%$ has been obtained.

(⁶) B. S. Sood: *Proc. Roy. Soc.*, A **247**, 375 (1958).

Recently VERONESI and co-workers [7] have measured the asymmetry ratio in the elastic scattering of polarized photons of energy 1.28 mc², and using the T matrix for the pure Rayleigh scattering, they find a good agreement with the theoretical
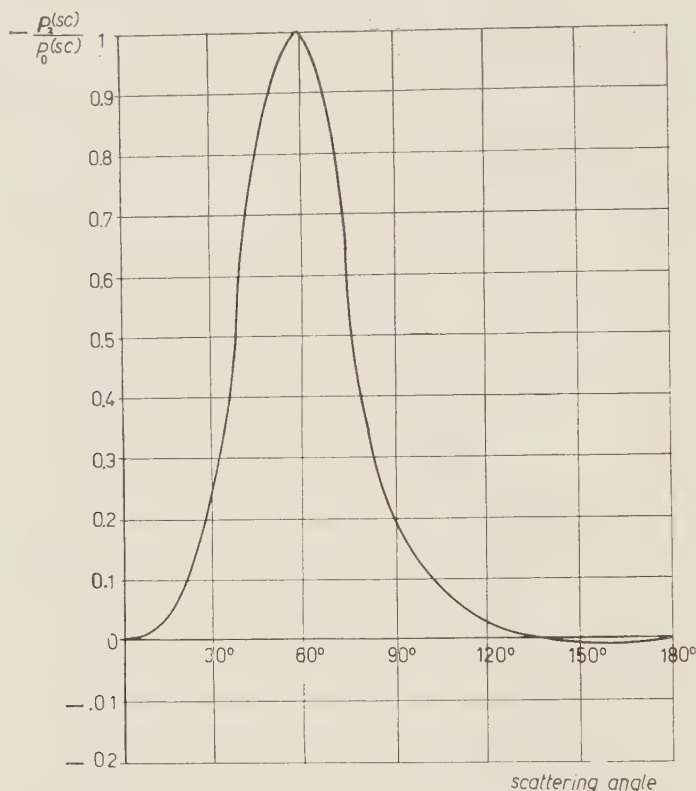
Fig. 3. – Degree of linear polarization of the scattered beam taking into account the interference of the Rayleigh and the Thomson scattering. $E\gamma = 2.56$ mc².

predictions of Brown and Mayers. This result is reasonable, because at this energy the interference contribution due to Thomson scattering is neglegible. Thomson scattering becomes important at 2.56 mc² and measurements at this energy are now been undertaken [8].

* * *

We wish to express our thanks to Prof. P. VERONESI for having put us this question and to Prof. A. BORSELLINO for his useful advice in this matter.

[7] D. BRINI, E. FUSCHINI, D. S. R. MURTY and B. VERONESI: *Nuovo Cimento*, **11**, 533 (1959).
[8] G. MANUZIO and S. VITALE: private communication.

Antihyperon Production in Antinucleon-Nucleon Collisions near threshold (*).

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Recently the production of hyperon-antihyperon pairs in nucleon-antinucleon collisions has become possible. Bearing this fact in mind, we have found it interesting to investigate the possibility of a theoretical estimation of the production ratios for different kinds of hyperon pairs. The present work is an improved version of the calculations quoted in [1] making use of some recent results on antinucleon-nucleon interactions.

Electromagnetic interactions will be neglected throughout the calculations. Consider the T-matrix element of the $\overline{\mathcal{N}}\mathcal{N} \to \overline{\mathcal{N}}\mathcal{N}$ scattering.

The experimental results show, that the absorptive part of T predominates, so in a reasonable approximation we can put

$$\langle f|T|i\rangle = \langle f|A|i\rangle .$$

The unitarity condition gives (using an obvious notation)

$$(1) \qquad \langle \overline{N}N|A|\overline{N}N\rangle = \sum_n \langle \overline{N}N|T|n\rangle\langle n|T^+|\overline{N}N\rangle .$$

Comparing this with the matrix element we are interested in

$$(2) \qquad \langle \overline{Y}Y|A|\overline{N}N\rangle = \sum_{n'} \langle \overline{Y}Y|T|n'\rangle\langle n'|T^+|N\overline{N}\rangle ,$$

(*) For a preliminary report on this work see: *Proc. of the Working Meeting on Physics of High Energy Particles*, Liblice (near Prague) (June 1958). (The paper quoted here contains some misprints).

[1] We describe this expansion for the case of pairs of like hyperons ($\overline{\Lambda}\Lambda$, $\overline{\Sigma}\Sigma$, $\overline{\Xi}\Xi$). The extension of the present estimation to the case of $\overline{\Lambda}\Sigma$ and $\overline{\Sigma}\Lambda$ is trivial.

it can be shown that in the neighbourhood of the $Y\overline{Y}$ threshold the same intermediate states play the most important role both in (2) and in (1). Thus if one considers the matrix element $\langle \overline{Y}Y | \Lambda | \overline{N}N \rangle = \langle Y\overline{Y} | T | \overline{N}N \rangle$ as a function of the hyperon mass (M_y), it can be expanded in a power series around the nucleon mass (M). With the approximation considered above, one finds (1):

(3) $$\langle \overline{Y}Y | T | \overline{N}N \rangle \approx \langle \overline{N}N | T | \overline{N}N \rangle + C_1(M_Y - M) + \dots .$$

The zero-order term is the matrix element of the — low energy — $\mathcal{N}\mathcal{N}$ scattering, and it is thus a slowly varying function of energy.

If we estimate the order of magnitude of the first-order term by putting $C_1 \sim \langle \overline{N}N | T | \overline{N}N \rangle | M$, we find, that the matrix element of the process in question is costant with an error of $(M_Y - M)/M$, and the transition probabilities are essentially determined by the phase space factors (i.e. the so-called statistical method can be applied).

In calculating the phase space factors we made further simplifications: we neglected the *production* of K-mesons (2).

The calculation of the statistical weights could be done in the standard manner (3).

The physical interpretation of the terms in the above expansion of the T-matrix suggested to choose as the interaction volume (in Fermi's terminology) a sphere with radius m_π^{-1} multipled by the corresponding Lorentz-contraction factor.

For the sake of convenience, we give the explicit form of the partial emission probability in the neighourhood of the threshold:

$$w = a[(2\tau + 4)^{\frac{1}{2}} - b]^{\frac{1}{2}} .$$

Here τ is the kinetic energy of the incident particle in the lab. system, measured in nucleon rest mass units, a depends on the interaction volume and the mass of the hyperon emitted; $b = M^{-1}(M_Y + M_{\overline{Y}})$, the values of a and b for different hyperon pairs are listed in the table.

TABLE I.

Product	$\overline{\Lambda}\Lambda$	$\overline{\Lambda}\Sigma, \overline{\Sigma}\Lambda$	$\overline{\Sigma}\Sigma$	$\overline{\Xi}\Xi$
$10^3 \cdot a$	1.74	1.58	1.33	0.69
b	2.38	2.46	2.54	2.82

For higher energies the expressions become more complicated; the results of the calculations are summarized in Fig. 1.

The cross section for the production is obtained by multiplying w by the corresponding total cross section.

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(2) According to Segrè [*Report at the 9th International Conference on High-Energy Physics* (Kiev, 1959)] the ratio of K-mesons emitted in $\mathcal{N}\mathcal{N}$ annihilation is about 3%.

(3) A report containing the details of the calculations will be published elsewhere.

Taking $\tau \approx 1$ (the antiproton energy of the Berkeley antihyperon experiments) and [3] $\sigma_t \approx 80$ mb, the cross section of $\overline{\Lambda}\Lambda$ production turns out to be

$$\sigma_{\overline{\Lambda}\Lambda} \approx 48 \text{ mb.}$$

The probability of the production of other antihyperon pairs practically equals zero.

Up to the writing of this paper 7 $\overline{\Lambda}\Lambda$ events and no other antihyperon pairs have been found. From these the experimental value of the cross-section was found to be

$$\sigma_{\overline{\Lambda}\Lambda} \approx 50 \text{ mb .}$$

Fig. 1. – Emission probabilities of different hyperon pairs as a function of τ, the kinetic energy of the incident particle. τ is measured in nucleon rest energy units.

The difference between experimental and theoretical data is within the statistical error.

* * *

We take pleasure in expressing our indebtedness to Prof. L. JÁNOSSY for his interest in the present work and to Dr. M. L. STEVENSON for having communicated to us the results of his experiments.
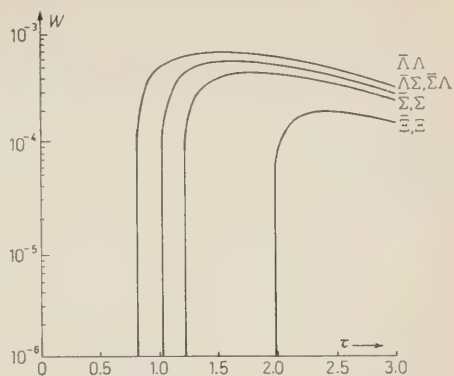
K-Meson Parity from Dispersion Relations.

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In recent months many efforts have been devoted to the study of the K-parity problem with the aid of dispersion relations [1]. However no definite conclusions have been reached yet. In this paper we want to rediscuss the problem, following the lines of the work by Amati [1] (hereafter referred to as A). This approach starts from an effective-range formula deduced from the dispersion relation for the forward K$^+$-proton scattering amplitude. The advantages of such a method are

    1) the formula used is weighted against the contributions coming from the unphysical region;

    2) it does not contain the hitherto rather ambiguously known real part of the forward K$^-$-proton scattering amplitude at zero kinetic energy, D$^-$(K);

    3) the obtained results are rather cut-off independent.

The dispersion relations used by other authors (see for example Karplus *et al.* [1]), are more convergent than those under discussion. However, under very general assumptions on the asymptotic behavior of the cross-sections it has been shown that the integrals used here are convergent [2]. Furthermore with rather general arguments it is possible to infer the effect of the very high energy contributions. We shall see that the advantages of this method shall allow us to get clearer conclusions than those obtained with other dispersion relation approaches.

As has been shown in A, if one makes the hypothesis of constant behaviour of the cross-section for K$^+$-proton scattering at low energy, for which good experimental information is lacking, there is evidence for a pseudoscalar K with respect to

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[1] P. T. Matthews and A. Salam: *Phys. Rev.*, **110**, 569 (1958); C. Goebel: *Phys. Rev.*, **110**, 572 (1958); K. Igi: *Progr. Theor. Phys.* (*Kyoto*), **19**, 238 (1958); D. Amati: *Phys. Rev.*, **113**, 1692 (1959); E. Galzenati and B. Vitale: *Phys. Rev.*, **113**, 1635 (1959); D. Amati, E. Galzenati and B. Vitale: *Nuovo Cimento*, **12**, 627 (1959); R. Karplus, L. Kerth and T. Kycia: *Phys. Rev. Lett.*, **2**, 510 (1959); M. M. Islam: *Nuovo Cimento*, **13**, 224 (1959).

[2] D. Amati, M. Fierz and V. Glaser: *Phys. Rev. Lett.*, **4**, 86 (1960)..

both Λ and Σ hyperons. In what follows we want to rediscuss the problem using the effective range formula deduced in A without the restrictive assumption of constant σ^- and adopting for the scattering amplitude in the unphysical region the solutions given by DALITZ and TUAN [3]. We will use the σ^+ slope in the low energy region as a parameter and obtain the K-meson parity with respect to it. A full discussion of the errors will also be carried out. Let w be the total laboratory energy of an incoming K-meson (the K-meson mass is taken as unit). Then the effective range formula is

(1) $$D^+(w) = D^+(1) + (w - 1)r^+(w) \, ,$$

$r^+(w)$ being given by

(2) $$B(w) = r^+(w) - \frac{1}{\pi} \int\limits_1^\infty \frac{k'\sigma^+(w')\,\mathrm{d}w'}{(w'-w)(w'-1)} + \frac{1}{\pi} \int\limits_1^\infty \frac{k'\sigma^-(w')\,\mathrm{d}w'}{(w'+w)(w'+1)} + U(w) \, ,$$

where $B(w) = R_\Lambda^+(w) + R_\Sigma^+(w)$ — see A for the explicit form — is the bound states contribution and has two different expressions according to whether the K-hyperon parity is even or odd. For example at the point $w = 1.22$, one has

(3) $$R_\Lambda^+(1.22) = \begin{bmatrix} -7.80 \\ 0.42 \end{bmatrix} \frac{g_\Lambda^2}{4\pi}; \qquad R_\Sigma^+(1.22) = \begin{bmatrix} -5.97 \\ 0.25 \end{bmatrix} \frac{g_\Sigma^2}{4\pi} \, ,$$

g_Λ and g_Σ being the K-hyperon-nucleon coupling constants. In R_Λ^+ the upper (lower) value must be taken if the K-Λ parity is even (odd). Similarly for R_Σ^+. Thus from the sign and magnitude of $B(w)$ it is possible to obtain information about the K-hyperon parities and coupling constants respectively. $U(w)$, unphysical region contribution, is given by

(4) $$U(w) = \frac{1}{\pi} \int\limits_{w_{\Lambda\pi}}^1 \frac{k'\sigma^-(w')\,\mathrm{d}w'}{(w'+w)(w'+1)} \, ,$$

with $w_{\Lambda\pi} \simeq 0.48$. As previously stated formula (2) does not contain $D^-(K)$. In the calculation of the first and the second integral on the r.h.s. of (2), which we will call I^+ and I^- respectively, we will use a cut-off of 5, corresponding to $T_{\mathrm{lab}}^{\mathrm{max}} \simeq 2$ GeV. However we shall discuss how our conclusions can be changed from the neglected parts of the integrals. Furthermore we will take $w = 1.22$ (as in A), corresponding to $T^{\mathrm{lab}} \simeq 110$ MeV. Of the two integrals I^+ and I^- only the former has singularities and these are confined to the low energy region. Thus a priori we can expect a rather strong dependence of I^+ from the slope of σ^+ in this region. To introduce the parameter which represents this slope we will use for σ^+ the expression

(5) $$\sigma^+(w') = \sigma^+(1) + b(w' - 1) \, ,$$

in the $1 < w' < 1.4$ region, while both for $\sigma^+(w')$ in the $w' > 1.4$ region and for $\sigma^-(w')$ we will use the experimental data. We note that the point $w' = 1.4$ lies

[3] R. H. DALITZ and S. F. TUAN: *Ann. Phys.*, **8**, 100 (1959).

in a region where three good experimental values for σ^+ have been obtained [4]. From these it is easy to obtain $\sigma^+(1.4) = (16 \pm 1)$ mb $= (10.0 \pm 0.6)K^{-2}$ which introduced in (5) permits the elimination of one of the two parameters $\sigma^+(1), b$. By elimination of $\sigma^+(1)$ one obtains

$$(6) \qquad \sigma^+(w') = (10.0 \pm 0.6) + b(w' - 1.4) .$$

Formula (5) is only the simplest way to introduce a parameter which represents the mean slope of σ^+ in the region $w' \leqslant 1.4$ and the obtained results are not very dependent upon the explicit w dependence of $\sigma^+(w)$. Only a pathological behaviour of σ^- with strong oscillations could make our results meaningless.

To make the discussion easier it is better to consider both integrals on the r.h.s. of (2) divided in two parts; from 1 to 1.4 and from 1.4 to 5. The b-dependent quantities in the expression that can be obtained from (2) taking $w = 1.22$ are $r^+(1.22)$ and $I^+ \big|_1^{1.4}$: Indeed from (1), using the experimental isotropy of the differential cross-section, the optical theorem and the fact that the K^+-proton potential is repulsive, which gives a minus sign to D^+, one can obtain

$$(7) \qquad r^+(1.22) = -4.55 \left[\frac{k_l}{k_b} \sqrt{4\pi\sigma^+(1.22) - k_b^2 \sigma^{+2}(1.22)} - \frac{N+K}{N} \sqrt{4\pi\sigma^+(1)} \right] .$$

Here k_l and k_b are, respectively, laboratory and baricentric momenta corresponding to $w = 1.22$; N and K are the nucleon and K-meson masses. Thus the b-dependent terms contributing to $B(1.22)$, which are obviously those containing σ^+, are given by

$$(8) \qquad f(b) = r^+(1.22) - \frac{1}{\pi} \cdot I^+ \bigg|_1^{1.4} .$$

Upon insertion in (8) of σ^+, as obtained by (6), one has

$$(9) \qquad f(b) \simeq -24.5 \left[\sqrt{8.32 - 0.12b} - \sqrt{10.0 - 0.4b} \right] + 17.8 - 0.91b .$$

In the approximation (5), (6), the errors on $f(b)$ come out only from $\sigma^+(1.4)$. They can easily be calculated using $\sigma^+(1.4) = 10.6 K^{-2}$ and $\sigma^+(1.4) = 9.4 K^{-2}$ instead of $\sigma^+(1.4) = 10.0 K^{-2}$.

Next let us discuss the b-independent part of the r.h.s. of (2). As we have already said, we have calculated the integrals $I^+ \big|_{1.4}^5$ and $I^- \big|_1^5$ taking the experimental values of the cross-sections; more precisely we have used the fit to the experimental points given by A. SALAM at the Kiev Conference [5]. The results were

$$(10) \qquad \frac{1}{\pi} I^+ \bigg|_{1.4}^5 = (18.2 \pm 0.7)K^{-2} ,$$

$$(11) \qquad \frac{1}{\pi} I^- \bigg|_1^5 = (6.4 \pm 0.6)K^{-2} .$$

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[4] L. T. KERTH, T. F. KYCIA and R. G. BAENDER: Bull. Am. Phys. Soc., 4, 25 (1959).
[5] A. SALAM: Kiev Conference Report (unpublished).

The errors indicated in (10) and (11) have been estimated as follows: for the error on I^+ we have taken 4% due to the fact that the value of σ^+ is known with about such a precision in every point. As for the error on I^- we have divided its origin into two parts: in the region $1 \leqslant w' \leqslant 1.4$ we have taken for σ^- the best fit to the three good experimental points existing in this region, which is essentially the fit given by Salam, and have estimated the errors from the extreme fits possible to these points. In the region $1.4 \leqslant w' \leqslant 5$ the function multiplying σ^- in the integral is very slowly varying. Thus what $I^-|_{1.4}^5$ is really sensitive to, is the mean value of $\sigma^-(w')$. We have estimated this mean value to be known with a precision of about 10%. In this way one obtains the value (11) for $I^-|_1^5$. Collecting the results (9), (10) and (11) we obtain from (2)

$$(12) \qquad B(1.22) = f(b) - (11.8 \pm 1.3) + U(1.22) .$$

There remains to be calculated the contribution given by the unphysical region term U. Theoretically four solutions are possible for the analytic extension of the imaginary part of the forward amplitude for K^- proton scattering in the unphysical region. The larger contribution comes from the curve indicated with $(a\text{-})$ in (3), which gives $U(1.22) \simeq 1.6K^{-2}$. The other solutions give smaller contributions. The bound-states contribution $B(1.22)$ as a function of b is drawn in Fig. 1
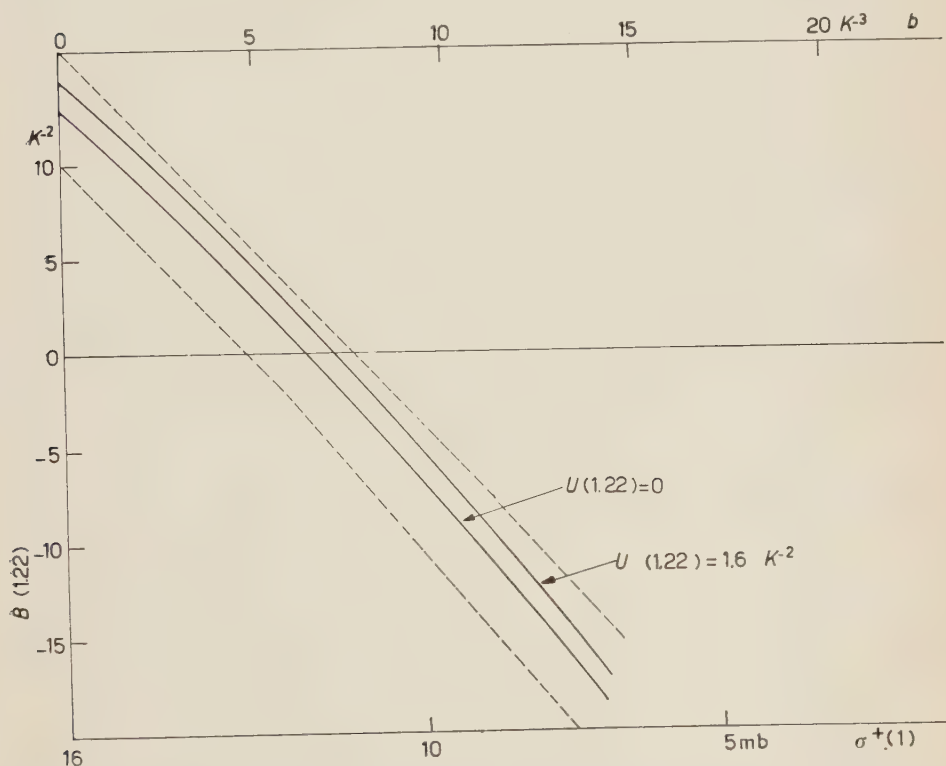

Fig. 1. – Continuous lines: bound states term $B(w)$ calculated at the point $w = 1.22\ K$ as a function of the slope b of σ^+ in the low energy region (higher; units K^{-3}) and of σ^+ (1) (lower; units mb) for two extreme values of the unphysical region term U. The dashed lines represent the errors on the curve with $U = 0$.

with $U(1.22)=0$. In Fig. 1 also the corresponding values of $\sigma^+(1)$ are given in abscissa. The broken lines give, for every b, the errors on $B(1.22)$. As it is easily seen $B(1.22)$ is positive (negative) for $b \geqslant 7$ ($b \leqslant 7$). Looking at formula (3) we see immediately that in the first (second) region the possibility of a scalar (pseudo-scalar) K-meson with respect to both Λ and Σ hyperons, is to be excluded.

Thus a future experimental determination of b will be decisive between these two possibilities. Taking for the unphysical region the solution $(a$-$)$ that gives $U(1.22)=1.6K^{-2}$ instead of $U(1.22)=0$, one has the effect of shifting the whole curve upwards. However, the new curve — see Fig. 1 — remains inside the experimental errors. Inclusion of other possible solutions gives for $B(1.22)$ curves which also lie nearer to the one with $U=0$. Thus we see that in any case our conclusions are rather independent of the unphysical region contributions. Furthermore, by inspection of (2), it is possible to understand the effect of including the high energy parts of the integrals. Beyond the cut-off 5 one easily sees that the denominator in I^- is about twice that in I^+ and that they tend to become equal with increasing energy. This is also the behaviour expected for the numerators, due to the fact that values of σ^- about twice those of σ^+ are suggested by the existing experimental points and that asymptotically the two cross-sections must become equal [6]. That σ^- remains higher as σ^+ above 2 GeV is, in our opinion, a rather plausible hypothesis, because such high energy cross-sections are expected to be predominantly inelastic and the number of open channels for K^-p inelastic scattering is larger than the same number for K^+p at the same energy (possibility of hyperon formation). Thus the two contributions should cancel in (2), showing the cut-off independence of the dispersion relations used.

In conclusion the results of this note may be considered to be the following:

The present evidence for a weak energy dependence of σ^+ at low energy is consistent only with a pseudo-scalar K meson (pseudo-scalar at least with respect to one of the hyperons). This conclusion is based on the fact that a drop of more than ~ 4 mb in σ^+ from its characteristic constant behaviour at $(100 \div 200)$ MeV, when the kinetic energy decreases down to zero, seems to be a rather unlikely possibility. To ensure our conclusion then, it would be extremely interesting to have a direct experimental determination of σ^+ in the very low energy region.

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[6] I. IA. POMERANČUK: *Žurn. Éksp. Teor. Fiz.*, **34**, 725 (1958) [translation: *Sov. Phys. Journ. Exsp. Theor. Phys.*, **7** (34), 499 (1958)].

A Possible Non-uniform Motion of a Free Particle if its Scalar Proper Field is Taken into Account.

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The classical relativistic equations of motion of a point particle subjected to external and proper scalar fields have been given by P. Havas [1] in the form

$$(1) \qquad m\dot{v}^\mu - \frac{g^2}{3}(\ddot{v}^\mu + v^\mu \dot{v}_\sigma \dot{v}^\sigma) - \frac{1}{2}g^2\chi^2 v^\mu +$$

$$+ g^2\chi^2 \int_{-\infty}^{\tau} \frac{\dot{s}^\mu}{s^2} J_2(\chi s)\, \mathrm{d}\tau' + g^2\chi \frac{\mathrm{d}}{\mathrm{d}\tau}\left[v^\mu \int_{-\infty}^{\tau} \frac{1}{s}J_1(\chi s)\, \mathrm{d}\tau'\right] = gF^\mu + g\frac{\mathrm{d}}{\mathrm{d}\tau}[vU]\,.$$

Here m is the interaction independent rest mass of the particle, g its mesonic charge, and χ is $\hbar^{-1}$ times the rest mass of the quantum of the scalar field obeying the Schrödinger-Gordon equation. The J's represent the Bessel functions of the first kind. The v^μ is the four velocity of the particle, τ its proper time. Dot means differentiation with respect to τ. Further

$$(2) \qquad\qquad s^\mu = z^\mu(\tau) - z^\mu(\tau')\,, \qquad\qquad s = +\,(s_\sigma s^\sigma)^{\frac{1}{2}}\,,$$

$z^\mu(\tau)$ being the world line of the particle, while $F^\mu = -\,\partial^\mu U$ represents the conservative external scalar field, if it exists at all.

There is no doubt that among the extremely rare ideas concerning the solution of the eq. (1) the most remarkable is that proposed by P. Havas [2]. According to this both the eq. (1) and the extra condition

$$v_\sigma v^\sigma = 1\,,$$

[1] P. Havas: *Phys. Rev.*, **87**, 309 (1952).
[2] P. Havas: *Phys. Rev.*, **93**, 882 (1954).

can be satisfied in the case of a free particle, identically vanishing U, by assuming a solution of the form

(3)
$$
\begin{cases}
z^0(\tau) = \dfrac{1}{A}\,\sinh A\tau\,, \\[3mm]
z^1(\tau) = \dfrac{1}{A}\,\cosh A\tau\,,
\end{cases}
$$

if and only if A is a root of the equation

(4)
$$
m\,A + g^2\chi^2 I_0\left(\frac{\chi}{A}\right) K_0\left(\frac{\chi}{A}\right) = 0\,,
$$

where I_0 and K_0 are the modified Bessel functions of the first and second kind, respectively.

For all the real values of the argument I_0 has positive value. Restricting ourselves to the real numbers, K_0 can have meaning for positive arguments only and in this case it has positive values everywhere [3]. Therefore it is evident that eq. (4) cannot have real roots.

If we would take for a moment $\chi = 0$, we would have

$$
m\dot{v}^\mu - \frac{g^2}{3}\,(\ddot{v}^\mu + v^\mu \dot{v}_\sigma \dot{v}^\sigma) = 0\,,
$$

which admits a self-accelerating solution of the form

$$
v^\mu(\tau) = A^\mu \exp\left[C\exp\left[3m\tau/g^2\right]\right] + B^\mu \exp\left[-C\exp\left[3m\tau/g^2\right]\right]\,,
$$

where A^μ, B^μ and C are the constants of integration subjected to the conditions

$$
A_\sigma A^\sigma = B_\sigma B^\sigma = 0\,, \qquad A_\sigma B^\sigma = \tfrac{1}{2}\,.
$$

Based on this statement we may hope that also in the case of non-vanishing χ the eq. (1) would admit runaway selfaccelerating solutions of a certain type for the free particle.

Since the eq. (4) can have only complex roots, it seems to be probable that runaway solutions of the type (3) are at the same time oscillating ones, the frequency of which is given by the imaginary part of the root of eq (4). So it may happen that the exact runaway solutions will be analogous in some respect to the « Zitterbewegung » of the free electron. Therefore it would be worth-while to investigate into the details of this question.

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[3] See *e.g.* N. W. Mc LACHLAN: *Bessel Functions for Engineers* (Oxford, 1948).

Dipole Ghost Contributions to Propagators.

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In Heisenberg's theory of the elementary particles [1] the field equation is completed by proposing the form

$$(1) \qquad \frac{1}{2} S'(p) = \int \varrho(\varkappa^2) \, \mathrm{d}\varkappa^2 \left[\frac{\gamma_\mu p_\mu + i\varkappa}{p^2 + \varkappa^2} - \frac{\gamma_\mu p_\mu + i\varkappa}{p^2} + \frac{\gamma_\mu p_\mu \varkappa^2}{(p^2)^2} \right],$$

for the Fourier transform of the vacuum expectation value of

$$S'(x, x') = \langle 0 | \{ \psi(x), \overline{\psi}(x') \} | 0 \rangle \, .$$

The form (1) requires a quantization by indefinite metric, the more detailed properties of which are defined just by (1). The second and third members together show some resemblances to terms corresponding to dipole ghost states [1,2] found explicitly in another theory, namely in the Lee model [3]. Indeed the two members in the singularity $p^2 = 0$ are equal to

$$(2) \qquad \lim_{\varepsilon \to 0} \gamma_\mu p_\mu \frac{\varkappa^2}{\varepsilon} \left(\frac{1}{p^2} - \frac{1}{p^2 + \varepsilon} \right),$$

which in the limit seems to refer to a contribution from a dipole ghost with zero rest mass.

Our aim is here to show that the relativistic generalization of a dipole ghost state, defined appropriately in analogy of that found in the Lee model, gives a similar but not identical result as stands in (1). For the sake of simplicity we treat the

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[1] W. Heisenberg: *Rev. Mod. Phys.*, **29**, 269 (1957); *Zeits. Naturfor.*, **14a**, 441 (1959), other references here.

[2] See especially W. Heisenberg: *Zeits. Phys.*, **144**, 1 (1956).

[3] W. Heisenberg: *Nucl. Phys.*, **4**, 532 (1957).

case of a simple hermitian scalar field φ. In connection with the indefinite metric the terminology and method of GUPTA ([4]) is used; for the field theoretical calculations the Källén-Lehmann technique is applied, according to which we always work in the Heisenberg picture.

We suppose there exists an energy-momentum operator P_μ with the properties:

$$(3) \qquad [P_\mu, \varphi] = i\,\partial_\mu \varphi\,, \qquad P_i = P_i^*\,, \qquad P_0 = P_0^*\,.$$

If P_μ has two one-particle eigenstates, one with negative norm (ghost state)

$$P_\mu |p\rangle = p_\mu |p\rangle\,, \qquad \langle p|p'\rangle = \delta_{pp'}\,, \qquad p_\mu^2 = -\mu^2\,, \qquad p_0 > 0\,,$$

$$P_\mu |p\rangle' = p_\mu |p\rangle'\,, \qquad {}'\langle p|p'\rangle' = -\delta_{pp'}\,, \qquad p_\mu^2 = -\varkappa^2\,, \qquad p_0 > 0\,,$$

the contribution of these states to Δ' is, of course

$$\Delta' \sim \Delta(\mu^2) - \Delta(\varkappa^2)\,.$$

Let us suppose now that by modifying some data in the theory these states overlap and P_μ possesses only one eigenvector (beside others, the contribution of which are not studied here) with zero norm:

$$(4) \qquad P_\mu |p\rangle = p_\mu |p\rangle\,, \qquad \langle p|p\rangle = 0\,, \qquad p_\mu^2 = -\mu^2\,, \qquad p_0 > 0\,.$$

This is allowed by the general principles of a theory with an indefinite metric ([5]) and it occurs indeed in the Lee model. But in this case the eigenvectors of P_μ do not form a complete system. One has to add for the completeness the dipole ghost state $|pD\rangle$ defined by

$$(5) \qquad P_\mu |pD\rangle = p_\mu |pD\rangle + C_\mu |p\rangle\,, \qquad \langle pD|pD\rangle = 0\,, \qquad p_\mu^2 = -\mu^2\,, \qquad p_0 > 0\,,$$

where because of relativistic invariance

$$(6) \qquad C_\mu = f(p^2)\,p_\mu\,.$$

(5) is the relativistic generalization of the definition of a dipole ghost state. $|p\rangle$ and $|pD\rangle$ are not orthogonal:

$$(7) \qquad \langle pD|p'\rangle = a\,\delta_{pp'}\,,$$

and since both states have zero norm, the values of a and f are rather arbitrary. From (5), (6) and (7)

$$\pi_\mu = \langle pD|P_\mu|pD\rangle = C_\mu a\,, \qquad \pi_i^* = \pi_i\,, \qquad \pi_0^* = \pi_0\,,$$

([4]) S. N. GUPTA: *Can. Journ. Phys.*, **35**, 961 (1957).

([5]) L. K. PANDIT: *Suppl. Nuovo Cimento*, **11**, 157 (1959).

and thus fa is real. The unit operator in this subspace is

$$(8) \qquad \sum \left\{ |pD\rangle \frac{1}{a^*} \langle p| + |p\rangle \frac{1}{a} \langle pD| \right\},$$

and the contribution to Δ'

$$(9) \qquad \Delta' = \sum \left\{ \langle 0|\varphi(x)|pD\rangle \frac{1}{a^*} \langle p|\varphi(x')|0\rangle + \langle 0|\varphi(x)|p\rangle \frac{1}{a} \langle pD|\varphi(x')|0\rangle \right\} - \sum \{x \rightleftarrows x'\}.$$

From (3) and (4)

$$(10) \qquad \langle 0|\varphi(x)|p\rangle = \langle 0|\varphi(0)|p\rangle \exp[ip_\mu x_\mu],$$

but from (3) and (5)

$$(11) \qquad \langle 0|\varphi(x)|pD\rangle = \{\langle 0|\varphi(0)|pD\rangle + iC_\mu x_\mu \langle 0|\varphi(0)|p\rangle\} \exp[ip_\mu x_\mu],$$

showing the peculiar x dependence [2,3]. Substituting (10) and (11) into (9) and requiring again relativistic invariance, one finds

$$(12) \qquad \Delta'(x - x') = c_1 \, \Delta(x - x') - c_2 (x_\mu - x'_\mu) \partial_\mu \, \Delta(x - x'),$$

where c_1 and c_2 are real, c_1 may be also negative, sign $c_2 = $ sign fa. From (12) for the Fourier transform it follows

$$(13) \qquad \Delta'(p) = (c_1 - 2c_2) \frac{1}{p^2 + \mu^2} - 2c_2 \frac{\mu^2}{(p^2 + \mu^2)^2},$$

i.e. the contribution of states with zero rest mass is the usual D with a non necessarily positive factor. The contribution of a dipole ghost alone can not be calculated, since the unit operator has the form (8). Nevertheless, substituting zero mass only into the denominator (*i.e.* for p^2 large), a similar expression results as in (1).

The discrepancy between (2) and (13) can be visualized as follows: (2) corresponds to a calculation where, essentially,

$$S' = \sum \langle 0|\varphi(x)|n\rangle \langle n|\varphi(x')|0\rangle - \sum (x \rightleftarrows x'),$$

was calculated first, when there were two energy eigenstates as yet, and only afterwards a limiting process was performed thus establishing a situation in which states of type (4) and (5) appear. In the above calculations one possesses first states of type (4) and (5) and only afterwards S' is calculated. These two procedures give different results, since in the state vector space in the limit a radical change takes place. If in a theory there are states of the type (4) and (5), S' has to be calculated by the method given above.

The form (1) of S', brings forth a special type of indefinite metric in the state vector space which may just as well be called a dipole ghost; however, we wished to show here that this form is not identical with the contribution originating from a Lee model type dipole ghost.

Relaxation of Dislocations in Copper.

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CORRIGENDA ET ADDENDA

At p. 303 of this paper the drawings of Figs. 20 and 21 have been unintentionally exchanged. This is rather evident from the text and the Authors take this occasion to emphasize that, as the comparison between the different types of spectra is of a basic significance in the theory of dislocation motion, the best approximation to the experimental results is given by a relaxation spectrum with *a constant value of the activation energy* W, and different values for the limiting frequencies ω_A.

LIBRI RICEVUTI E RECENSIONI

J. HAMILTON – *The Theory of Elementary Particles*. Oxford at the Clarendon Press, 1959. 75 sh, pp. IX + 482.

Lo studio delle particelle elementari è oggi certamente il campo più aperto della fisica, nel quale malgrado i salti in avanti della teoria, benchè grandi, sembra che siano ancora molto distanti da un completamento come quello conseguito in anni ormai lontani dalla fisica classica. In questa fretta di avanzare si ha talvolta l'impressione di essere male informati sulle posizioni raggiunte, in quanto l'informazione è affidata ad un grande numero di voluminose riviste; e nelle riviste, come si sa, c'è di tutto, sia le pietre miliari che i sassi. Ben viene perciò un libro chiaro che raccoglie i fondamenti, e riassume i punti di vista ormai ben consolidati.

Questo volume di HAMILTON ha una pregevole struttura sotto vari aspetti. Prima di tutto non è così astratto da costringere ad un complicato lavoro di particolarizzazione per arrivare ad un calcolo « con i numeri »; poi, è ricchissimo di esempi, i quali, da soli, formerebbero forse già un'utile monografia; infine è dotato di una bibliografia così aggiornata da far pensare istintivamente alle materiali difficoltà incontrate dall'editore e dall'autore nel corso della stampa.

Se si può fare un paragone, il testo somiglia sotto molti punti di vista al ben noto HEITLER anche se difficilmente si può prevedere per esso una vita altrettanto lunga.

A mio parere i capitoli VII e VIII (Selection rules e Polarization analysis rispettivamente) costituiscono la parte più originale del volume, se confrontato con altri dello stesso genere. Viceversa le 11 pagine sulle relazioni di dispersione alla fine del capitolo VI sembrano essere un'augusta dimora per un tale argomento (tre pagine sono utilizzate per richiamare le formule di Kramers e Kroning; va però detto che è difficilissimo reperire nella letteratura la dimostrazione di queste relazioni ormai più che trentenni).

Ottima la veste tipografica e ragionevole il prezzo.

C. BERNARDINI

H. A. HANS and R. B. ADLER – *Circuit Theory of Linear Noisy Networks*. The Technology Press of M.I.T. and John Wiley and Sons Company, New York, 1959, pp. XII + 79.

Questo volumetto fa parte di una collana di monografie, pubblicate con un certa periodicità a cura del M.I.T., che riguardano tutte studi originali su problemi di ricerca avanzata. Tali studi risultano in genere troppo estesi per costituire l'argomento di un articolo da rivista, nè per altro hanno l'aspetto di un libro compiuto in se stesso considerando il carattere d'avanguardia dei risultati esposti.

Potrebbero essere considerati quasi dei quaderni di laboratorio riordinati in una versione discorsiva e conseguente: è quanto si può affermare di questo libro che studia le proprietà del rumore di fondo nelle reti lineari.

Gli autori fanno notare che la cifra di rumore F, universalmente adottata per caratterizzare il rumore di un amplificatore, è semplicemente un numero definito operativamente e non una quantità fisica dedotta da postulati chiaramente stabiliti o da leggi della natura. Tale cifra di rumore non costituisce una caratteristica intrinseca del rumore dell'amplificatore, ma dipende ad esempio dalla impedenza della sorgente connessa ai morsetti d'entrata, dalle interconnessioni fra stadio e stadio, come pure dal grado di reazione introdotto in ogni stadio. Lo scopo della nuova ricerca degli autori consiste invece nel trovare una misura delle caratteristiche di rumore che non includa il guadagno esplicitamente e che possa essere minimizzata nei confronti dei parametri esterni all'amplificatore stesso e sia invariante per trasformazioni di rete che non introducano perdite, come ad esempio l'introduzione di una maglia di reazione senza perdite.

A questo punto lo studio si allarga per comprendere oltre gli amplificatori anche le reti a più coppie di terminali. Si cercano le proprietà invarianti delle caratteristiche di rumore di queste reti e gli Autori, usando il metodo matriciale, sviluppano una forma « canonica » da cui è possibile ricavare tutti gli invarianti in maniera semplice. Viene quindi data una interpretazione del significato fisico di questi invarianti in termini di « potenza disponibile » (available power).

Nell'ultimo capitolo, la teoria generale è applicata al caso degli amplificatori a due coppie di terminali per stabilire un limite minimo sulle sue caratteristiche di rumore.

Lo studio è indubbiamente molto interessante sia per l'originalità dei risultati raggiunti sia per l'interesse che attualmente hanno gli amplificatori a basso rumore. Tuttavia occorre aggiungere che il criterio di determinare il limite inferiore nel rumore di un amplificatore è utile, non tanto per progettarne uno che raggiunga tale minimo, ma per stabilire fino a che punto un dato progetto, che deve prima soddisfare tanti altri requisiti pratici, si avvicina alle sue migliori caratteristiche di rumore. In questo senso i criteri esposti nel libro costituiscono una nuova ed efficace guida nel definire le caratteristiche di rumore degli amplificatori.

U. PELLEGRINI

A. MESSIAH - *Méchanique Quantique*. Dunod, Paris, 1960, Tome II, pag. XV-974.

Abbiamo già recensito a parte il primo volume di questo trattato di meccanica quantistica sapientemente compilato da MESSIAH sulla base delle sue lezioni a Saclay. Il secondo volume, ora uscito, completa l'opera che viene così a porsi tra le più moderne e dettagliate trattazioni oggi esistenti di meccanica quantistica.

Già nel recensire il primo volume del trattato di MESSIAH avemmo modo di notare due aspetti positivi del lavoro che s'imponevano al lettore: l'equilibrio tra la parte formale e la parte interpretativa, e la novità, sul piano didattico, di alcuni punti dell'esposizione. Il secondo volume va lodato per gli stessi pregi, con l'aggiunta di una particolare lode per avere l'autore fatto posto, in una trattazione universitaria di meccanica quantistica, ad una estesa ed aggiornata discussione delle proprietà di simmetria, quale raramente si trova in lavori del genere.

Questo secondo volume contiene i capitoli dal tredicesimo al ventunesimo e due lunghe appendici.

Il capitolo tredicesimo sui momenti angolari presenta una sistematica trattazione dell'argomento a partire dalla definizione degli operatori di momento angolare, dallo studio delle loro rappresentazioni e dallo studio degli operatori di rotazione, dei momenti angolari intrinseci, dei teoremi di addizione e dei coefficienti di Clebsch-Gordon, sino ai coefficienti di Racah, ed all'algebra degli operatori tensoriali irriducibili col teorema di Wigner-Eckart.

Il capitolo quattordicesimo è dedicato ai sistemi di particelle identiche ed al principio di Pauli. Viene sviluppata l'algebra degli operatori di simmetrizzazione ed esposta anche l'algebra dello spin isotopico per sistemi di nucleoni.

Il capitolo quindicesimo tratta della proprietà di invarianza e dei teoremi di conservazione, ed, in particolare, della inversione di tempo. Vengono dapprima dati dei complementi matematici sugli operatori antilineari e sui gruppi di trasformazione. Quindi viene definita l'operazione di inversione di tempo in meccanica classica ed in meccanica quantistica e formulato il principio di reversibilità microscopica.

Il capitolo sedicesimo, sui metodi di approssimazione per perturbazioni stazionarie, sviluppa il calcolo delle perturbazioni per livelli non degeneri e per livelli degeneri. L'ultima parte del capitolo riproduce lo sviluppo a tutti gli ordini mediante il metodo della risolvente, recentemente formulati da KATO ed un metodo simile dovuto a BLOCH.

Il capitolo diciassettesimo contiene la teoria delle perturbazioni dipendenti dal tempo e l'esposizione dei metodi adiabatici ed impulsivi. I metodi variazionali sono trattati nel capitolo diciottesimo insieme alla loro applicazione agli atomi ed alle molecole.

Il capitolo diciannovesimo tratta la teoria della diffusione. Questo capitolo, il quindicesimo capitolo e le due appendici costituiscono, a nostro avviso, le parti più interessanti e didatticamente

originali di questo secondo volume. La discussione dei problemi d'urto è basata sull'uso delle funzioni di Green. Il capitolo contiene anche una presentazione dei metodi variazionali per il calcolo delle fasi ed una discussione delle proprietà generali della matrice S che seguono dalla unitarietà e dalle proprietà di invarianza, inclusa l'inversione di tempo.

Concludono l'opera i capitoli ventesimo e ventunesimo, rispettivamente sulla equazione di Dirac e sulla teoria della radiazione. Questi ultimi due capitoli sono certamente scritti con sufficiente rigore ma lontani dal livello di originalità dei capitoli precedenti. Il loro studio può riuscire sufficiente a completare la cultura di meccanica quantistica di un fisico sperimentale, ma è naturalmente insufficiente per la preparazione di un fisico teorico.

Infine vanno menzionate le due appendici, la prima sui coefficienti d'addizione e matrici di rotazione, estesa fino alla discussione dei simboli $9j$, la seconda sulla teoria dei gruppi, che rappresenta in sè una pregevole ed essenziale esposizione della teoria.

Indubbiamente i due volumi di questa « Méchanique Quantique » di MESSIAH offrono nell'insieme un testo adeguato e moderno della meccanica quantistica di particelle. Non si può dire, e del resto nessuno lo pretenderebbe, che la lettura degli ultimi due capitoli dell'opera sia sufficiente per lo studio della teoria dei campi. A questo proposito vorremmo riportare la conclusione, a prima vista disorientante, cui ci ha spinto l'esame di questa opera. Due volumi, uno di quasi cinquecento pagine, l'altro di quasi mille pagine, servono ad esporre nozioni ed applicazioni, a prima vista tutte importanti ed al livello sufficientemente generale e non specializzato dei nostri corsi universitari. Questi due volumi d'altra parte non esauriscono certamente l'insieme di nozioni teoriche necessarie ad un fisico: si pensi alla meccanica anali-

tica, alla elettrodinamica classica, alla relatività, alla meccanica statistica, ecc. Non menzioniamo qui la teoria dei campi, o la fisica nucleare teorica che possono essere ritenute materie di specializzazione. Inoltre stiamo solo parlando di materie teoriche. È chiaro che la preparazione di un fisico richiede sempre maggiori sforzi, maggiore studio e certamente più tempo di quanto non ne richiedeva in passato. Non vogliamo qui, fuori dalla giusta sede, indicare o proporre soluzioni per tali problemi. Vale però la pena di far notare questo ampliamento della materia inevitabile in un campo di studio in costante evoluzione e le cui applicazioni vanno incessantemente moltiplicandosi.

Per concludere questa recensione vogliamo riportare due delle appropriate citazioni che l'autore pone agli inizi dei vari capitoli. All'inizio della parte sulla meccanica quantistica relativistica si trova riportata una citazione, in francese, dal Cantico dei Cantici, « Je suis noire, et pourtant je suis belle... », che indubbiamente induce a meditare. All'inizio del capitolo sulla teoria dei campi sta scritto, questa volta nella lingua di Dante e di Machiavelli: « Se non è vero, è bene trovato ».

R. GATTO

Progress in Nuclear Physics, Vol. 7, 1959. Editore: O. R. FRISCH. Pergamon Press, Condon, pag. VII 325.

Sei articoli compongono questo settimo volume della annuale rassegna di fisica nucleare diretta da O. Frisch. Si tratta di sei articoli su argomenti di grande interesse ed attualità. Il primo articolo è di D. V. BUGG sulle camere a bolle. L'articolo contiene una esposizione delle teorie sul funzionamento delle camere a bolle, una rassegna sui

diversi tipi di liquidi, una descrizione dei metodi di illuminazione e di fotografia, una descrizione dei magneti usati per camere a bolle, ed infine una descrizione dei metodi di analisi dei fotogrammi col dovuto rilievo all'utilissimo apparecchio automatico per proiezione e scanning che viene usato a Berkely, che va sotto il terrificante nome Frankenstein. Il secondo articolo di F. R. METZGER sulla fluorescenza di risonanza nei nuclei contiene una accurata esposizione del fenomeno estesamente studiato da otto anni a questa parte. Vengono esaminate in dettaglio le varie condizioni di sorgente, dividendo le sorgenti in due gruppi, quelle per cui la vita media del livello emittente è molto maggiore del tempo per collisione del nucleo di rinculo con gli atomi vicini, e quelle per cui vale la situazione opposta. Segue un articolo di B. G. HARVEY sulla « spallation », un termine molto usato recentemente per indicare quelle reazioni in cui il nucleo colpito emette un certo numero di frammenti. Si tratta di fenomeni complicati e di non semplice spiegazione. Tuttavia metodi statistici, per la prima volta sviluppati da GOLDBERGER e da BERNARDINI, BOOTH e LINDEBAUM, hanno recentemente permesso, con l'ausilio di calcolatori elettronici, una descrizione abbastanza soddisfacente. L'articolo di A. E. GLASSGOLD sul modello ottico per diffusioni nucleari costituisce una semplice ed accurata rassegna del metodo, che recentemente ha avuto uno sviluppo sorprendente. Particolare menzione merita l'articolo di L. GRODZINS di Brookhaven sulla misura della elicità. Ci è parso notevole ed in certo senso impressionante come, a relativamente poca distanza dalla scoperta della non conservazione della parità, si siano sviluppate e rapidamente raffinate le tecniche sperimentali per la misura della polarizzazione longitudinale dei fotoni e dei leptoni, al punto da richiedere un completo articolo di rassegna per la loro

descrizione, e per la esposizione dei risultati. L'ultimo articolo, teorico, dovuto a J. J. SAKURAI, sulle interazioni deboli, contiene una istruttiva rassegna sui recenti sviluppo della teoria delle interazioni deboli, che si spinge fino alla discussione dei problemi attualmente insoluti nel tentativo di stabilire una interazione universale.

<div align="right">R. GATTO</div>

MARTIN SCHWARZSCHILD - *Structure and Evolution of the Stars*. Un volume in 8 di XVII-296 pagine con 51 figure e 39 tavole. Princeton University Press, Princeton, N.J., 1958. Prezzo rilegato dollari 6.00.

La recente introduzione dei risultati della fisica nucleare nella astrofisica teorica insieme al sempre maggiore numero di misure eseguite negli osservatori astronomici, hanno dato in questi ultimi anni un apporto sensibile agli studi che riguardano la struttura e l'evoluzione delle stelle.

Nuovi elementi e nuove idee in questo campo sono raccolti ed esposti molto organicamente nell'ottimo libro di SCHWARZSCHILD.

Esso è diviso in otto capitoli: dopo una breve rassegna dei più importanti dati di osservazione, l'A., nel II capitolo, passa alle leggi fisiche che, stando alle attuali conoscenze, devono legare le grandezze meccaniche e termodinamiche caratteristiche di una stella. Le equazioni di queste leggi vengono risolte per lo più numericamente; e nel III capitolo vi è una succinta esposizione dei metodi seguiti.

Nel seguito l'A., quasi utilizzando queste premesse, passa a descrivere quantitativamente i diversi modelli stellari che permettono di rappresentare la struttura di ciascuna singola stella a seconda della massa, della età e della composizione iniziale di essa (cap. IV,

V, VI, VII). Un sommario dei risultati e delle conferme della teoria, alla quale l'A. ha contribuito in buona parte con precedenti lavori, chiude l'opera.

Si tratta, in sintesi, di una rassegna molto aggiornata dell'argomento; essa è indirizzata prevalentemente a quanti lavorano in questo campo specializzato e, più in generale, a matematici o fisici di interessi affini. A tutti riuscirà assai utile la bibliografia che è particolarmente accurata. Ma una categoria ancora più vasta di lettori potrà accedere con utilità e diletto a questo libro, per la comprensione del quale ci sembra sufficiente qualche cognizione, al livello divulgativo, sull'argomento, associata ad una conveniente dimestichezza con la matematica nel biennio di ingegneria.

<div align="right">R. CERVELLATI</div>

J. S. ALLEN – *The neutrino*, pp. 168, figg. 64, tab. 9 (Princeton University Press, Princeton, N. J., 1958, $ 4.50).

Nella interessante ed utilmente didattica collezione della Princeton U. P. è apparsa ultimamente una monografia sul neutrino, dovuta ad uno dei più attivi cultori dell'argomento: J. S. ALLEN.

Potrà sembrare strano che a più di trent'anni di distanza dalla geniale intuizione di Pauli e dall'altrettanto geniale applicazione fattane da FERMI nella teoria del decadimento β, questo sia il primo libro (a parte qualche breve ed infrequente articolo riassuntivo) completamente dedicato ad una particella che ha avuto ed ha tuttora tanta importanza nello svolgimento della teoria delle particelle elementari. Ciò è dovuto forse al fatto che, mentre dal punto di vista fenomenologico la sua posizione è chiaramente definita, a causa delle sue inafferrabili qualità la sua individuazione è risultata di difficoltà estrema.

Forse proprio per questa ragione il libro risulta interessante: perchè espone essenzialmente le diverse tecniche messe in opera per ricavare, per via sperimentale, le varie caratteristiche di questa fondamentale particella.

Nel libro gli sviluppi teorici sono piuttosto circoscritti e si limitano ad esporre i fatti fondamentali necessari alla comprensione del copioso materiale sperimentale. La maggior parte della esposizione riguarda i fenomeni cosiddetti di bassa energia: sette sugli otto capitoli del libro, sono ad essi dedicati. La parte sostanziale si riferisce alla descrizione delle esperienze sulle correlazioni angolari elettrone-neutrino, comprendendo anche le recenti ricerche sulla non conservazione della parità nel decadimento β, ed alle misure della sua massa. Un capitolo è dedicato alla rivelazione del neutrino libero ed uno al problema dell'esatta descrizione del neutrino, come particella di Dirac oppure di Majorana, ricavabile dalle esperienze sul doppio decadimento β. Minore estensione è data invece al trattamento sui decadimenti mesonici, ed in generale gli ultimi sviluppi, come del resto è comprensibile in un campo che è in rapida evoluzione, sono trattati piuttosto frettolosamente.

Ad ogni modo è da segnalare la esposizione chiara, dettagliata ed esauriente delle numerose esperienze eseguite in queste difficili ricerche e se ciò, da un lato, può costituire motivo di interessante e piacevole lettura per il fisico sperimentale, può lasciare, d'altra parte, insoddisfatto il teorico più esigente.

G. POIANI